SDM5008 Advanced Control for Robotics

Lecture 7: Rigid Body Dynamics

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Outline

- ~~Spatial Acceleration~~ ✓
- Spatial Force (Wrench)
- Spatial Momentum
- Newton Euler Equation in Spatial Algebra ✓

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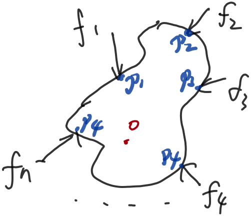
- Spatial Force (Wrench)
- Spatial Momentum

Spatial Force (Wrench)

- Consider a rigid body with many forces on it and fix an arbitrary point O in space



- The net effect of these forces can be expressed as
  - A force f, acting along a line passing through O

    $$f = \sum_i f_i$$

  - A moment n_O about point O

    $$n_O = \sum_{i=1}^{n} \vec{op_i} \times f_i$$

- **Spatial Force (Wrench):** is given by the 6D vector

$$\mathcal{F} = \begin{bmatrix} n_O \\ f \end{bmatrix}$$

wrench

Recall: for twist

$$v_q = v_o + \omega \times \vec{oq}$$

· what if we change reference point from "o" to "q": be definition.

$$n_q = \sum_i \vec{qp_i} \times f_i = n_o + \sum_i (\vec{qp_i} - \vec{op_i}) \times f_i = n_o + \sum_i \vec{qo} \times f_i$$

$$= n_o + \vec{qo} \times f$$

$$= n_o + f \times \vec{oq}$$

$$\boxed{\begin{aligned} n_q &= n_o + f \times \vec{oq} \\ &= n_o + \vec{qo} \times f \end{aligned}}$$

coordinate-free

Spatial Force in Plücker Coordinate Systems

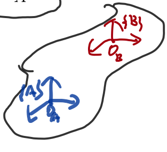
- Given a frame $\{A\}$, the Plücker coordinate of a spatial force $\mathcal{F}$ is given by

convention: choose frame origin as reference point for moment.

$$ {}^A\mathcal{F} = \begin{bmatrix} {}^A n_{O_A} \\ {}^A f \end{bmatrix} $$

For twist
$$ {}^A\mathcal{V} = {}^A X_B \, {}^B\mathcal{V} $$

- Coordinate transform: $\underbrace{{}^A\mathcal{F}}_{6\times1} = \underbrace{{}^A X_B^*}_{6\times6} \underbrace{{}^B\mathcal{F}}_{6\times1}$ where ${}^A X_B^* = {}^B X_A{}^T$



Frame $\{A\}$, $\{B\}$ with ${}^A T_B = ({}^A R_B, {}^A p_B)$

$$ {}^A\mathcal{F} = \begin{bmatrix} {}^A n_{O_A} \\ {}^A f \end{bmatrix}, \quad {}^B\mathcal{F} = \begin{bmatrix} {}^B n_{O_B} \\ {}^B f \end{bmatrix} $$

- ${}^A f = {}^A R_B \, {}^B f \quad \dots \quad ①$

- moment. Coordinate-free: $n_{O_A} = n_{O_B} + f \times (\overrightarrow{O_B O_A})$

choose $\{A\}$-frame to express: ${}^A n_{O_A} = {}^A R_B \, {}^B n_{O_B} + {}^A R_B \left({}^B f \times \underbrace{{}^B(\overrightarrow{O_B O_A})}_{{}^B p_A} \right)$

$= {}^A R_B \left({}^B n_{O_B} + (-{}^B p_A \times {}^B f) \right) \quad \dots \quad ②$

$$ ①, ② \Rightarrow \begin{bmatrix} {}^A n_{O_A} \\ {}^A f \end{bmatrix} = \underbrace{\begin{bmatrix} {}^A R_B & -{}^A R_B [{}^B p_A] \\ 0 & {}^A R_B \end{bmatrix}}_{{}^A X_B^*} \begin{bmatrix} {}^B n_{O_B} \\ {}^B f \end{bmatrix} $$

Recall: ${}^B X_A = \begin{bmatrix} {}^B R_A & 0 \\ [{}^B p_A] \, {}^B R_A & {}^B R_A \end{bmatrix} \Rightarrow {}^A X_B^* = {}^B X_A{}^T$

Wrench-Twist Pair and Power

$[a]^T = -[a]$

f, v power $= f^T v = \langle f, v \rangle$

- Recall that for a point mass with linear velocity v and linear force f. Then we know that the power (instantaneous work done by f) is given by $f \cdot v = f^T v$

- This relation can be generalized to spatial force (i.e. wrench) and spatial velocity (i.e. twist)

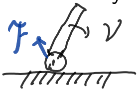
- Suppose a rigid body has a twist ${}^A\mathcal{V} = ({}^A\omega, {}^A v_{o_A})$ and a wrench ${}^A\mathcal{F} = ({}^A n_{o_A}, {}^A f)$ acts on the body. Then the power is simply

$$\underbrace{P}_{\text{scalar power}} = \underbrace{({}^A\mathcal{V})^T}_{1\times 6} \underbrace{{}^A\mathcal{F}}_{6\times 1} = {}^A\mathcal{F}^T\, {}^A\mathcal{V} = \langle {}^A\mathcal{F}, {}^A\mathcal{V} \rangle$$

$$= \underbrace{{}^A\omega^T\, {}^A n_{o_A}}_{\text{rotational power}} + \underbrace{{}^A v_{o_A}^T\, {}^A f}_{\text{linear power}}$$

Joint Torque

- Consider a link attached to a 1-dof joint (e.g. revolute or prismatic). Let $\hat{\mathcal{S}}$ be the screw axis of the joint. The velocity of the link induced by joint motion is given by: $\mathcal{V} = \hat{\mathcal{S}}\dot{\theta}$



$\mathcal{V} = \hat{\mathcal{S}}\dot{\theta}$

- $\mathcal{F}$ be the wrench provided by the joint. Then the power produced by the joint is

$$P = \mathcal{V}^T\mathcal{F} = \underbrace{(\hat{\mathcal{S}}^T\mathcal{F})}_{\text{scalar}}\underbrace{\dot{\theta}}_{\text{scalar}} \triangleq \tau\dot{\theta}$$

$\tau \triangleq \hat{\mathcal{S}}^T\mathcal{F} = \mathcal{F}^T\hat{\mathcal{S}}$

- $\tau = \hat{\mathcal{S}}^T\mathcal{F} = \mathcal{F}^T\hat{\mathcal{S}}$ is the projection of the wrench onto the screw axis, i.e. the effective part of the wrench.

- Often times, τ is referred to as joint "torque" or generalized force

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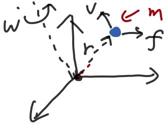
- Spatial Force (Wrench)
- Spatial Momentum ✓

Rotational Inertia (1/2)



- Recall momentum for point mass:

Linear motion		rotational motion
velocity: $v = \dot{r}$, $a = \dot{v} = \ddot{r} \in \mathbb{R}^3$	$\longleftrightarrow$	$\omega = \hat{\omega}\dot{\theta}$, $v = \omega \times r$
force: $f = ma = m\dot{v} = m\ddot{r}$	$\longleftrightarrow$	moment. $n = r \times f$
Linear momentum: $\underset{3\times1}{L} = m \underset{3\times1}{v}$	$\longleftrightarrow$	Angular momentum

$$\underset{3\times1}{\phi} = r \times L$$

$$= r \times (m v)$$

$$= r \times (m\, \omega \times r)$$

$$= \underbrace{\underset{\text{scalar}}{m}\, \underset{3\times3}{[r]}\, \underset{3\times3}{[-r]}}_{\text{Inertia matrix}}\, \underset{3\times1}{\omega}$$

$\phi = I\,\omega$

Rotational Inertia (2/2) $\in$ of rigid body

$\phi = \sum_i m_i [r_i][r_i]^T \omega$

$\rho(r)dr$

"this matrix depends on coordinate system"

- Rotational Inertia: $\bar{I} = \int_V \rho(r)[r][r]^T dr$
  - $\rho(\cdot)$ is the density function of the body
  - $\bar{I}$ depends on coordinate system

    $\bar{I}_c = \sum_i m_i [\overrightarrow{cr_i}][\overrightarrow{cr_i}]^T$

  - It is a constant matrix if the origin coincides with CoM

What's def of Center of Mass

$$c \triangleq CoM \triangleq \frac{1}{m}\int \rho(r) r dr \approx \frac{1}{m}\sum m_i r_i$$

If q is CoM, then $\frac{1}{m}\sum m_i (\overrightarrow{qr_i}) = 0$ ....

$$\Rightarrow \sum m_i \underbrace{(\overrightarrow{qr_i})}_{3\times 1} = 0 \Rightarrow \sum m_i [\overrightarrow{qr_i}] = 0 \quad \text{---} \; ①$$

Spatial Momentum

$a \times b = b \times (-a)$

- Consider a "rigid body" with spatial velocity $\mathcal{V}_C = (\omega, v_C)$ expressed at the center of mass C ← (derivation below works only when $C =$ com)
  - Linear momentum:

    $L \triangleq m v_C$ ← velocity of CoM ; why? $L \triangleq \sum_i m_i v_i = \sum_i m_i (v_c + \omega \times \vec{Cr_i})$

    $= \sum_i m_i v_c + \sum_i m_i \, \vec{Cr_i} \times (-\omega)$ ← 3×1

    $= m v_c + \left(\sum_i m_i \cancel{[\vec{Cr_i}]}\right) \cdot (-\omega)$ ← 3×1

    0 (because C is com)

  - Angular momentum about CoM: $\phi_c = \bar{I}_c \omega$

    $\phi_c \triangleq \sum_i \vec{Cr_i} \times (m_i v_i) = \sum_i \vec{Cr_i} \times (m_i v_c + m_i \omega \times \vec{Cr_i})$

    $= \left(\sum_i (\vec{Cr_i}) \times m_i v_c\right) + \sum_i m_i \vec{Cr_i} \times (-\vec{Cr_i}) \times \omega$

    $= \left(\sum_i m_i \cancel{[\vec{Cr_i}]}\right) v_c + \bar{I}_c \omega$

    0

  - Angular momentum about a point O:

    $\phi_O = \sum_i \vec{Or_i} \times (m_i v_i) \xrightarrow{\text{V.F.T.}} \phi_c + \vec{OC} \times L$

- Spatial Momentum:

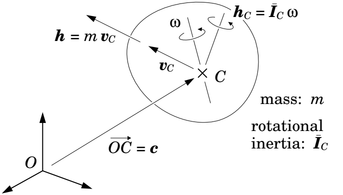
  relation

  $$\phi_o = \phi_c + \vec{OC} \times L$$

  ← coordinate free

$$h = \begin{bmatrix} \phi_c \\ L \end{bmatrix}$$

ϕ_c ← reference point ↓ choose to be origin of coordinate system



Change Reference Frame for Momentum

- Spatial momentum transforms in the same way as spatial forces:

$$^{A}h = {}^{A}X_{C}^{*}\,{}^{C}h$$

Spatial Inertia

- Inertia of a rigid body defines linear relationship between velocity and momentum.

- Spacial inertia $\mathcal{I}$ is the one such that

$$h = \mathcal{I}\mathcal{V}$$

- Let {C} be a frame whose origin coincide with CoM. Then

$${}^{C}\mathcal{I} = \begin{bmatrix} {}^{C}\bar{I}_c & 0 \\ 0 & mI_3 \end{bmatrix}$$

Spatial Inertia

- Spatial inertia wrt another frame {A}:

$$ {}^{A}\mathcal{I} = {}^{A}X_{C}^{*}\,{}^{C}\mathcal{I}\,{}^{C}X_{A} $$

- Special case: ${}^{A}R_{C} = I_3$

More Discussions

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