

MEE5114 Advanced Control for Robotics

Lecture 6: Velocity Kinematics: Geometric and Analytic Jacobian of Open Chain

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CLEAR Lab

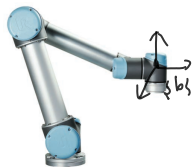
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Outline

- Background
- Geometric Jacobian Derivations
- Analytic Jacobian

Velocity Kinematics



FK: Find the func of $T_b(\theta_1, \dots, \theta_n)$

$$\theta_1, \theta_2, \dots, \theta_n \longrightarrow T_b(\theta_1, \dots, \theta_n)$$

- **Velocity Kinematics:** How does the velocity of $\{b\}$ relate to the joint velocities $\dot{\theta}_1, \dots, \dot{\theta}_n$
- This depends on how to represent $\{b\}$'s velocity
 - Twist representation \rightarrow **Geometric Jacobian**

$v_b = \begin{bmatrix} w \\ v \end{bmatrix}$, $v_b(\theta, \dot{\theta})$: it turns out v_b is a linear func of $\dot{\theta}$

$$\Rightarrow v_b(\theta, \dot{\theta}) = J(\theta) \dot{\theta}$$

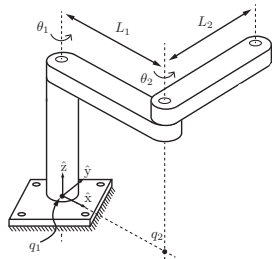
- Local coordinate of SE(3) \rightarrow **Analytic Jacobian**

$\underbrace{J(\theta)}_{\text{Geometric } J}$

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Simple Illustration Example: Geometric Jacobian (1/2)



Simple Illustration Example: Geometric Jacobian (2/2)

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Geometric Jacobian: General Case (1/3)

- Let $\mathcal{V} = (\omega, v)$ be the end-effector twist (coordinate-free notation), we aim to find $J(\theta)$ such that

$$\mathcal{V} = J(\theta)\dot{\theta} = J_1(\theta)\dot{\theta}_1 + \cdots + J_n(\theta)\dot{\theta}_n$$

- The i th column $J_i(\theta)$ is the end-effector *velocity* when the robot is rotating about \mathcal{S}_i at unit speed $\dot{\theta}_i = 1$ while all other joints do not move (i.e. $\dot{\theta}_j = 0$ for $j \neq i$).
- Therefore, in **coordinate free** notation, J_i is just the screw axis of joint i :

$$J_i(\theta) = \mathcal{S}_i(\theta)$$

Geometric Jacobian: General Case (2/3)

- The actual coordinate of \mathcal{S}_i depends on θ as well as the reference frame.
- The simplest way to write Jacobian is to use local coordinate:

$${}^i J_i = {}^i S_i, \quad i = 1, \dots, n$$

- In fixed frame $\{0\}$, we have

$${}^0 J_i(\theta) = {}^0 X_i(\theta) {}^i S_i, \quad i = 1, \dots, n \quad (1)$$

- Recall: ${}^0 X_i$ is the change of coordinate matrix for spatial velocities.
- Assume $\theta = (\theta_1, \dots, \theta_n)$, then

$${}^0 T_i(\theta) = e^{[{}^0 \bar{S}_1] \theta_1} \dots e^{[{}^0 \bar{S}_i] \theta_i} M \quad \Rightarrow \quad {}^0 X_i(\theta) = [\text{Ad}_{{}^0 T_i(\theta)}] \quad (2)$$

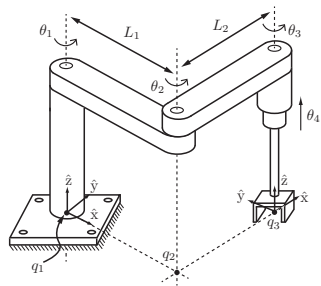
Geometric Jacobian: General Case (3/3)

- The Jacobian formula (1) with (2) is conceptually simple, but can be cumbersome for calculation. We now derive a recursive Jacobian formula
- Note: ${}^0J_i(\theta) = {}^0S_i(\theta)$
 - For $i = 1$, ${}^0S_1(\theta) = {}^0S_1(0) = {}^0\bar{S}_1$ (independent of θ)
 - For $i = 2$, ${}^0S_2(\theta) = {}^0S_1(\theta_1) = \left[\text{Ad}_{\hat{T}(\theta_1)} \right] {}^0\bar{S}_2$, where $\hat{T}(\theta_1) \triangleq e^{[{}^0\bar{S}_1]\theta_1}$
 - For general i , we have

$${}^0J_i(\theta) = {}^0S_i(\theta) = \left[\text{Ad}_{\hat{T}(\theta_1, \dots, \theta_{i-1})} \right] {}^0\bar{S}_i \quad (3)$$

where $\hat{T}(\theta_1, \dots, \theta_{i-1}) \triangleq e^{[{}^0\bar{S}_1]\theta_1} \dots e^{[{}^0\bar{S}_{i-1}]\theta_{i-1}}$

Geometric Jacobian Example



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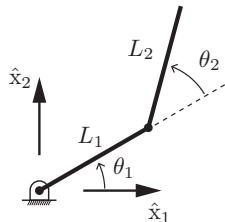
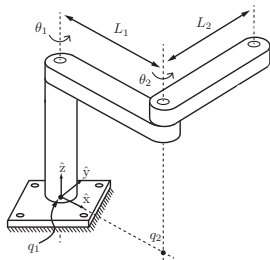
Analytic Jacobian

- Let $x \in \mathbb{R}^p$ be the task space variable of interest with desired reference x_d
 - E.g.: x can be Cartesian + Euler angle of end-effector frame
 - $p < 6$ is allowed, which means a partial parameterization of SE(3), e.g. we only care about the position or the orientation of the end-effector frame
- Analytic Jacobian: $\dot{x} = J_a(\theta)\dot{\theta}$
- Recall Geometric Jacobian: $\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} = J(\theta)\dot{\theta}$
- They are related by:

$$J_a(\theta) = E(x)J(\theta) = E(\theta)J(\theta)$$

- $E(x)$ can be easily found with given parameterization x

Simple Illustration Example: Analytic Jacobian (1/3)



$$\begin{cases} {}^0p_{b,x} = L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2) \\ {}^0p_{b,y} = L_1 \sin(\theta_1) + L_2 \sin(\theta_1 + \theta_2) \\ {}^0p_{b,z} = L_0 \end{cases}$$

Simple Illustration Example: Analytic Jacobian (2/3)

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Simple Illustration Example: Analytic Jacobian (3/3)

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More Discussions

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