MEE5114 Advanced Control for Robotics

Lecture 6: Velocity Kinematics: Geometric and Analytic Jacobian of Open Chain

Prof. Wei Zhang

CLEAR Lab

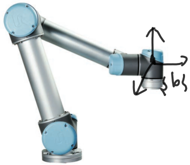
Southern University of Science and Technology, Shenzhen, China

https://www.wzhanglab.site/

Outline

- Background
- Geometric Jacobian Derivations
- Analytic Jacobian

Velocity Kinematics



FK: Find the func of $T_b(\theta_1, \cdots, \theta_n)$

$$\theta_1, \theta_2, \cdots, \theta_n \longmapsto T_b(\theta_1, \cdots, \theta_n)$$

- **Velocity Kinematics**: How does the velocity of $\{b\}$ relate to the joint velocities $\dot{\theta}_1, \ldots, \dot{\theta}_n$
- This depends on how to represent $\{b\}$'s velocity
  - Twist representation $\to$ **Geometric Jacobian**

$\mathcal{V}_b = \begin{bmatrix} w \\ v \end{bmatrix}$, $\mathcal{V}_b(\theta, \dot{\theta})$: it turns out. $\mathcal{V}_b$ is a linear func of $\dot{\theta}$

$$\Rightarrow \mathcal{V}_b(\theta, \dot{\theta}) = J(\theta)\, \dot{\theta}$$

Geometric Jacobian

$6 \times n$

  - Local coordinate of SE(3) $\to$ **Analytic Jacobian**

$$\theta_1, \theta_2 \cdots, \theta_n \xrightarrow{F.K} T_b(\theta_1, \theta_2 \cdots \theta_n) = (R, p)$$

x, y, z

RPY: α, β, γ

$$\longrightarrow \underbrace{\begin{bmatrix} p_x \\ p_y \\ p_z \\ \alpha \\ \beta \\ \gamma \end{bmatrix}}_{} \in \mathbb{R}^6$$

Outline

. e.g. $x = \begin{bmatrix} p_x \\ p_y \\ p_z \\ \alpha \\ \beta \\ \gamma \end{bmatrix}$ ①, or $x = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$ ②, or $x = \begin{bmatrix} p_y \\ \beta \end{bmatrix}$ ③

task space variable $\leftarrow$ x: local coordinate, depends on θ.

- Background

$\theta \xmapsto{g(\cdot)} x \qquad x = g(\theta_1, \cdots, \theta_n)$

$\dot{x} = \underbrace{\left[\frac{\partial g}{\partial \theta}\right]}_{\text{Jacobian matrix}} \begin{bmatrix} \dot\theta_1 \\ \vdots \\ \dot\theta_n \end{bmatrix}$

Jacobian matrix . ① : $6 \times n$

② : $3 \times n$

③ : $2 \times n$

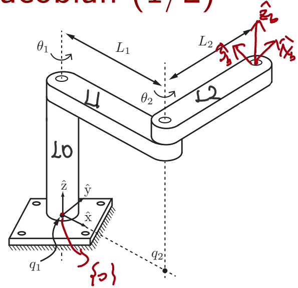
- Geometric Jacobian Derivations

- Analytic Jacobian

$g = \begin{bmatrix} g_1(\theta_1, \cdots, \theta_n) \\ \vdots \\ g_m(\theta_1, \cdots, \theta_n) \end{bmatrix}$

$\frac{\partial g}{\partial \theta} = \left[\frac{\partial g_i}{\partial \theta_j}\right]_{i,j}$

Simple Illustration Example: Geometric Jacobian (1/2)



- Goal: $\mathcal{V}_b(\theta_1, \theta_2, \dot\theta_1, \dot\theta_2)$

– Coordinate-free

- Screw axis: Joint 1: S_1 ; Joint 2: $S_2(\theta_1)$

– Spatial velocity of each link (when $\dot\theta_1, \dot\theta_2 \neq 0$)

Link 0: $\mathcal{V}_{L0} = 0 \in \mathbb{R}^6$; $\mathcal{V}_{L1} = S_1\dot\theta_1$

Link 2: $\mathcal{V}_{L2} = \mathcal{V}_{L2/L1} + \mathcal{V}_{L1/L0} = S_2(\theta_1)\cdot\dot\theta_2 + S_1\dot\theta_1$

$\mathcal{V}_{L2/L0}$

Goal: $\mathcal{V}_b = \mathcal{V}_{L2} = S_1\dot\theta_1 + S_2(\theta_1)\dot\theta_2 = \underbrace{\overbrace{[S_1 \,\vdots\, S_2(\theta_1)]}^{\text{Geometric Jacobian}}}_{6\times 2}\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\end{bmatrix}$

$J_i(\theta)$: i^{th} column of Geometric Jacobian $= [J_1(\theta) \,\vdots\, J_2(\theta)]\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\end{bmatrix}$

Simple Illustration Example: Geometric Jacobian (2/2)

- $J_i(\theta)$: twist of body $\{b\}$ when $\dot{\theta}_i = 1$, , $\theta_j = 0$, $j \neq i$

– Computation: Let's work with $\{0\}$, ${}^0S_1(\theta) = {}^0S_1(\theta=0) = {}^0\bar{S}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

${}^0S_2(\theta_1)$

· Let $\theta_1 = 0$, ${}^0S_2(0) = {}^0\bar{S}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -L_1 \\ 0 \end{bmatrix}$

· $\theta_1 \neq 0$, ${}^0\bar{S}_2 = {}^0S_2(0) \xrightarrow{\hat{T}(\theta_1) = e^{[{}^0\bar{S}_1]\theta_1}} {}^0S_2(\theta_1) = \underbrace{\left[Ad_{\hat{T}_1(\theta_1)}\right]}_{6 \times 6} \cdot {}^0\bar{S}_2$

$${}^0J(\theta) = \left[{}^0\bar{S}_1 \;\vdots\; \left[Ad_{\hat{T}(\theta_1)}\right] {}^0\bar{S}_2 \right]$$

Geometric Jacobian: General Case (1/3)

- Let $\mathcal{V} = (\omega, v)$ be the end-effector twist (coordinate-free notation), we aim to find $J(\theta)$ such that

$$\mathcal{V} = J(\theta)\dot{\theta} = J_1(\theta)\dot{\theta}_1 + \cdots + J_n(\theta)\dot{\theta}_n$$

$$= \left[J_1(\theta) \,\vdots\, J_2(\theta) \,\vdots\, \cdots \,\vdots\, J_n(\theta) \right] \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}$$

- The ith column $J_i(\theta)$ is the end-effector *velocity* when the robot is rotating about $\mathcal{S}_i$ at unit speed $\dot{\theta}_i = 1$ while all other joints do not move (i.e. $\dot{\theta}_j = 0$ for $j \neq i$).

- Therefore, in **coordinate free** notation, J_i is just the screw axis of joint i:

$$J_i(\theta) = \mathcal{S}_i(\theta)$$

Geometric Jacobian: General Case (2/3)

- The actual coordinate of $\mathcal{S}_i$ depends on θ as well as the reference frame.

- The simplest way to write Jacobian is to use local coordinate:

$$ {}^iJ_i = {}^iS_i, \quad i = 1, \ldots, n $$

$\mathcal{V}_b = [J_1(\theta) \;\cdots\; J_n(\theta)] \begin{bmatrix} \vdots \\ \vdots \end{bmatrix}$

common reference frame

- In fixed frame $\{0\}$, we have

$$ {}^0J_i(\theta) = {}^0X_i(\theta)\, {}^iS_i, \quad i = 1, \ldots, n \tag{1} $$

- Recall: ${}^0X_i$ is the change of coordinate matrix for spatial velocities.

- Assume $\theta = (\theta_1, \ldots, \theta_n)$, then

$$ {}^0T_i(\theta) = e^{[{}^0\bar{\mathcal{S}}_1]\theta_1} \cdots e^{[{}^0\bar{\mathcal{S}}_i]\theta_i} M \quad \Rightarrow \quad {}^0X_i(\theta) = \left[\mathrm{Ad}_{{}^0T_i(\theta)}\right] \tag{2} $$

Geometric Jacobian: General Case (3/3)

- The Jacobian formula (1) with (2) is conceptually simple, but can be cumbersome for calculation. We now derive a $\boxed{\text{recursive Jacobian formula}}$

- Note: ${}^0J_i(\theta) = {}^0S_i(\theta)$
  - For $i = 1$, ${}^0S_1(\theta) = {}^0S_1(0) = {}^0\bar{\mathcal{S}}_1$ (independent of θ)
  - For $i = 2$, ${}^0S_2(\theta) = {}^0S_1(\theta_1) = \left[\mathrm{Ad}_{\hat{T}(\theta_1)}\right] {}^0\bar{\mathcal{S}}_2$, where $\hat{T}(\theta_1) \triangleq e^{[{}^0\bar{\mathcal{S}}_1]\theta_1}$

$\hat{T}_1(\theta_1, \theta_2) = e^{[\hat{S}]\alpha}$

$i = 3$, ${}^0S_3(\theta) = {}^0S_3(\theta_1, \theta_2)$

$\hat{T}(\theta_1, \theta_2) = e^{[{}^0\bar{S}_1]\theta_1} e^{[{}^0\bar{S}_2]\theta_2}$

${}^0\bar{S}_3 = {}^0S_3(0, 0) \longrightarrow \underbrace{\left[\mathrm{Ad}_{\hat{T}(\theta_1, \theta_2)}\right]}_{6\times 6} {}^0\bar{S}_3$

  - For general i, we have

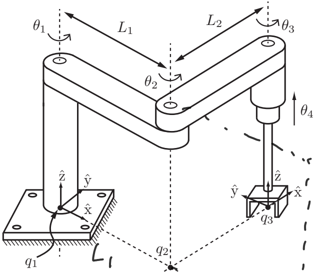
$$
{}^0J_i(\theta) = {}^0S_i(\theta) = \left[\mathrm{Ad}_{\hat{T}(\theta_1,\ldots,\theta_{i-1})}\right] {}^0\bar{\mathcal{S}}_i \tag{3}
$$

$$
\text{where} \quad \hat{T}(\theta_1, \ldots, \theta_{i-1}) \triangleq e^{[{}^0\bar{\mathcal{S}}_1]\theta_1} \cdots e^{[{}^0\bar{\mathcal{S}}_{i-1}]\theta_{i-1}}
$$

$J(\theta) = \left[{}^0\bar{S}_1 \;\vdots\; [\mathrm{Ad}_{\hat{T}(\theta_1)}]\, {}^0\bar{S}_2 \;\vdots\; [\mathrm{Ad}_{\hat{T}(\theta_1,\theta_2)}]\, {}^0\bar{S}_3 \;\vdots\; [\mathrm{Ad}_{\hat{T}(\theta_1,\theta_2,\theta_3)}]\, {}^0\bar{S}_4 \;\vdots\; \cdots \;\vdots\; \cdots \right]$

Geometric Jacobian Example

- $J(\theta) = \left[S_1(\theta) \,\vdots\, S_2(\theta) \,\vdots\, S_3(\theta) \,\vdots\, S_4(\theta) \right]$ ← expressed in {0} frame



L_1, L_2, q_3

1°: find screw axis at home position ($\theta_1=\theta_2=\theta_3=\theta_4=0$)

$${}^0\bar{S}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad {}^0\bar{S}_1 = \begin{bmatrix} {}^0w_1 \\ {}^0v_1 \end{bmatrix}$$

$${}^0\bar{S}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -L_1 \\ 0 \end{bmatrix} \qquad {}^0v_2 = h\hat{s} - {}^0w_2 \times {}^0q_2 = 0 - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -L_1 \\ 0 \end{bmatrix}$$

$${}^0\bar{S}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -L_1-L_2 \\ 0 \end{bmatrix} \qquad {}^0v_3 = -{}^0w_3 \times q_3 = \begin{bmatrix} 0 \\ -L_1-L_2 \\ 0 \end{bmatrix}$$

$${}^0\bar{S}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad {}^0w_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad {}^0v_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\hat{T}_3 = e^{[{}^0\bar{S}_1]\theta_1} e^{[{}^0\bar{S}_2]\theta_2} e^{[{}^0\bar{S}_3]\theta_3}$$

2°: $${}^0J(\theta) = \left[{}^0\bar{S}_1 \,\vdots\, [Ad_{\hat{T}_1}]\,{}^0\bar{S}_2 \,\vdots\, [Ad_{\hat{T}_2}]\,{}^0\bar{S}_3 \,\vdots\, [Ad_{\hat{T}_3}]\,{}^0\bar{S}_4 \right]$$

$$\hat{T}_1 = e^{[{}^0\bar{S}_1]\theta_1} \qquad \hat{T}_2 = e^{[{}^0\bar{S}_1]\theta_1} e^{[{}^0\bar{S}_2]\theta_2}$$

$${}^0\mathcal{V}_b = {}^0J(\theta)\,\dot{\theta}$$

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- Analytic Jacobian

Analytic Jacobian

 $x=\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $x=\begin{bmatrix} a \\ b \end{bmatrix}$, $x = \rho \sin\theta$ (ρ, θ)

- Let $x \in \mathbb{R}^p$ be the task space variable of interest with desired reference x_d
  - E.g.: x can be Cartesian + Euler angle of end-effector frame
    spherical ← ZYZ, ZYX, (α, β, γ)
  - $p < 6$ is allowed, which means a partial parameterization of SE(3), e.g. we only care about the position or the orientation of the end-effector frame

$\Rightarrow x = g(\theta)$

- Analytic Jacobian: $\dot{x} = J_a(\theta)\dot{\theta}$ ← joint velocity

$\dot{x} = \left[\frac{\partial g}{\partial \theta}\right] \cdot \dot{\theta}$

- Recall Geometric Jacobian: $\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} = J(\theta)\dot{\theta}$

- They are related by:

$$J_a(\theta) = E(x)J(\theta) = E(\theta)J(\theta)$$

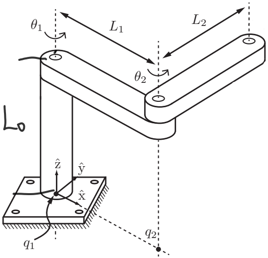
  - $E(x)$ can be ~~easily~~ found with given parameterization x

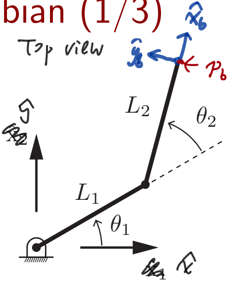
Simple Illustration Example: Analytic Jacobian (1/3)

- For example, let task space variable

$x = {}^0p_b$

$${}^0\dot{p}_b = \underbrace{\left[\frac{\partial g}{\partial \theta}\right]}_{\text{Analytic Jacobian } J_a(\theta)} \dot{\theta}$$





$${}^0p_b = \begin{cases} {}^0p_{b,x} = L_1\cos(\theta_1) + L_2\cos(\theta_1+\theta_2) \leftarrow g_1 \\ {}^0p_{b,y} = L_1\sin(\theta_1) + L_2\sin(\theta_1+\theta_2) \leftarrow g_2 \\ {}^0p_{b,z} = L_0 \leftarrow g_3 \end{cases}$$

$$= g(\theta_1, \theta_2) = \begin{bmatrix} g_1(\theta_1,\theta_2) \\ g_2(\theta_1,\theta_2) \\ g_3(\theta_1,\theta_2) \end{bmatrix}$$

$$\Rightarrow J_a(\theta) = \begin{bmatrix} \frac{\partial g_1}{\partial \theta_1} & \frac{\partial g_1}{\partial \theta_2} \\ \frac{\partial g_2}{\partial \theta_1} & \frac{\partial g_2}{\partial \theta_2} \\ \frac{\partial g_3}{\partial \theta_1} & \frac{\partial g_3}{\partial \theta_2} \end{bmatrix}$$

$$= \begin{bmatrix} -L_1\sin\theta_1 - L_2\sin(\theta_1+\theta_2) & -L_2\sin(\theta_1+\theta_2) \\ L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2) & L_2\cos(\theta_1+\theta_2 \\ 0 & 0 \end{bmatrix}$$

$J_a(\theta)$

Simple Illustration Example: Analytic Jacobian (2/3)

- $J_a(\theta) = E(\theta)\,\underline{J(\theta)}$

- Let ${}^0J(\theta)$ be geometric Jacobian, ${}^0\mathcal{V}_b = {}^0J(\theta)\begin{bmatrix}\dot\theta_1 \\ \dot\theta_2\end{bmatrix}$

$$\downarrow$$

$${}^0\mathcal{V}_b = \begin{bmatrix}{}^0\omega \\ {}^0v_0\end{bmatrix} \qquad S_1 = \begin{bmatrix}\omega_1 \\ {}^0v_1\end{bmatrix}$$

$$\underline{{}^0\dot{p}_b} = {}^0v_0 + {}^0\omega \times {}^0p_b = \underbrace{-{}^0p_b}_{[{}^0p_b]} \times {}^0\omega + {}^0v_0 = \left[-[{}^0p_b] \;\vdots\; I_{3\times3}\right]\begin{bmatrix}{}^0\omega \\ {}^0v_0\end{bmatrix}$$

$$= \underbrace{\left[-[{}^0p_b]^{(\theta)} \quad I_{3\times3}\right]{}^0J(\theta)}_{{}^0J_a(\theta)}\begin{bmatrix}\dot\theta_1 \\ \dot\theta_2\end{bmatrix}$$

Simple Illustration Example: Analytic Jacobian (3/3)

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More Discussions

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