SDM5008 Advanced Control for Robotics

Lecture 5: Product of Exponential and Kinematics of Open Chain

Prof. Wei Zhang

CLEAR Lab
Southern University of Science and Technology, Shenzhen, China
`https://www.wzhanglab.site/`

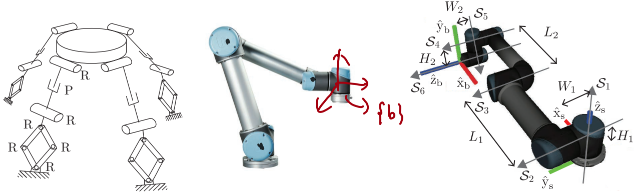
Outline

- Motivating Example
- Product of Exponential Formula Derivations
- Practice Example

Kinematics

robot: Multiple rigid bodies interconnected through joints

Kinematics is a branch of classical mechanics that describes the motion of points, bodies (objects), and systems of bodies (groups of objects) without considering the mass of each or the forces that caused the motion



- **Forward Kinematics**: calculation of the configuration (pose) $T = (R, p)$ of the end-effector frame {b} from joint variables $\theta = (\theta_1, \ldots, \theta_n)$

  $T(\theta_1, \theta_2, \cdots, \theta_n)$

- **Velocity Kinematics (Next Lecture)**: Deriving the Jacobian matrix: linearized map from the joint velocities $\dot{\theta}$ to the spatial velocity $\mathcal{V}$ of the end-effector

  $\mathcal{V}_b(\theta_1, \theta_2 . \theta_n)$

Illustration Example (1/3)

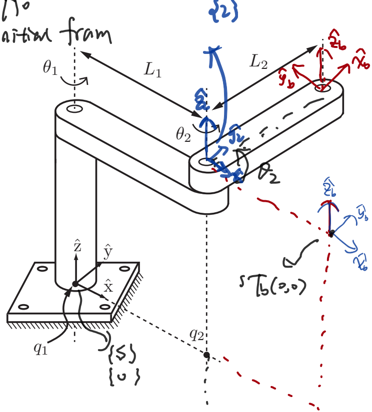
$A' = e^{[S_2]\theta_2} \cdot A_0$

initial fram

$S_2, \to \theta$

Consider a 2R robot

- Three links and two joints θ_1, θ_2
- Link/body frame attached to link i at joint i (one of possible choices)
- Fixed/world frame {s} frame , end-effector frame {b}
- **Goal**: compute ${}^sT_b(\theta_1, \theta_2)$: function of θ_1, θ_2
- Initial pose: $M \triangleq {}^sT_b(0, 0) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & L_1+L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$



$S_2 \leftrightarrow (\hat{s}, h, q_2)$

$${}^2S_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Illustration Example (2/3)

- Fix joint 1 at $\theta_1 = 0$, rotate joint 2 by θ_2, we have ${}^sT_b(0, \theta_2)$

- Rigid body motion for Link 2/body 2 ({b}), represented by screw motion with screw axis S_2, In coordinate free manner,

$$\text{Initial} \quad \underset{\theta_1=0,\ \theta_2=0}{M} \xrightarrow{\hat{T} = e^{[S_2]\theta_2}} [\hat{T}]\cdot M = e^{[S_2]\theta_2}\cdot M \Leftarrow T_b(0,\theta_2)$$

- In {s}-frame ({0})

$$ {}^sT_b(0,\theta_2) = e^{[{}^0S_2]\theta_2}\, \underbrace{{}^sT_b(0,0)}_{M}$$

· Notation: ${}^0S_2$ is a function θ_1, more precisely, should be written as ${}^0S_2(\theta_1)$ $\leftarrow \theta_1 = 1$

- For now, $\theta_1 = 0$, define ${}^0\bar{S}_2 = {}^0S_2(0)$

· ${}^0\bar{S}_2 = \begin{bmatrix} {}^0w_2 \\ {}^0v_2 \end{bmatrix}$, ${}^0w_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, ${}^0v_2 = \cancel{h\,{}^0\hat{s}} - {}^0w_2 \times {}^0q_2 = -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -L_1 \\ 0 \end{bmatrix}$

${}^0\hat{S} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \leftarrow (\hat{s}, h, q)$

${}^0\bar{S}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -L_1 \\ 0 \end{bmatrix}$

$\Rightarrow {}^sT_b(0,\theta_2) = e^{[{}^0\bar{S}_2]\theta_2} M$

Illustration Example (3/3)

- Fix joint 2 at θ_2, and rotate joint 1 by $\theta_1 \Rightarrow {}^sT_b(\theta_1, \theta_2)$

· Joint 1 screw axis S_1, S_1: independent of θ_1, θ_2,

define ${}^o\bar{S}_1 = {}^oS_1(\theta_1=0, \theta_2=0)$ ${}^o\bar{S}_1 = \begin{bmatrix} {}^o\omega_1 \\ {}^ov_1 \end{bmatrix}$, ${}^o\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

· ${}^o\bar{S}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$${}^ov_1 = h\,{}^o\hat{s}_1 - {}^o\omega_1 \times {}^oq_1$$
$$= -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

· Step 2: Initial pose: ${}^sT_b(0, \theta_2) \xrightarrow{\hat{T}_1 = e^{[{}^o\bar{S}_1]\theta_1}} {}^sT_b(\theta_1, \theta_2) = e^{[{}^o\bar{S}_1]\theta_1}\, {}^sT_b(0, \theta_2)$

$$\Rightarrow {}^sT_b(\theta_1, \theta_2) = \underbrace{e^{[{}^o\bar{S}_1]\theta_1} e^{[{}^o\bar{S}_2]\theta_2}}_{\text{PoE}} \cdot M$$

Outline

- Motivating Example
- Product of Exponential Formula Derivations
- Practice Example

Notation Setup (1/2) multi-body link ⟺ body

- Suppose that the robot has n joints and n links. Each joint has one degree of freedom represented by joint variable θ_i, $i = 1, \ldots, n$
  - θ_i: the joint angle (Revolute joint) or joint displacement (Primatic joint)

- Specify a fixed frame $\overset{\{b\}}{\{s\}}$: also referred to as frame $\{0\}$

frame $\{i\}$ moves with body i

- Attach frame $\{i\}$ to link i at joint i, for $i = 1, \ldots, n$

- Attach frame $\{b\}$ at the end-effector: sometimes referred to as frame {n+1}

constant (independent of $\theta_1, \theta_2 \ldots \theta_n$)

- ${}^{i}\mathcal{S}_i$: screw axis of joint i expressed in frame {i}

- ${}^{0}\mathcal{S}_i$: screw axis of joint i expressed in fixed frame $\{0\}$ (i.e. frame $\{s\}$)

${}^{0}S_i(\theta_1, \theta_2, \cdots, \theta_i)$

Notation Setup (2/2)

- For simplicity, we write configuration as T_{sb}, which is the same as ${}^sT_b$. Similarly, $T_{ij} = {}^iT_j$

- Note: ${}^i\mathcal{S}_i$ does not change when the robot moves (i.e. when θ changes), but ${}^0\mathcal{S}_i$ depends on $\theta_1, \ldots, \theta_i$. Sometimes, we write out the dependency explicitly, i.e. ${}^0\mathcal{S}_i(\theta_1, \ldots, \theta_i)$

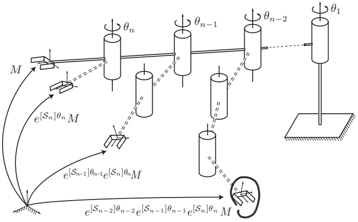
- Define home position: $\theta_1 = 0, \ldots, \theta_n = 0$. This is the configuration when all the joint angles are zero. One can also choose other *fixed* angles as the home position

constant

- Define ${}^0\bar{\mathcal{S}}_i = {}^0\mathcal{S}_i(0, \ldots, 0)$: the screw axis of joint i expressed in frame $\{0\}$, when the robot is at the home position.

Product of Exponential: Main Idea

- **Goal:** Derive $T_{sb}(\theta_1, \ldots, \theta_n)$
- Compute $M \triangleq T_{sb}(0, \ldots, 0)$: the configuration of end-effector when the robot is at home position



- Apply screw motion to joint n: $T_{sb}(0, \ldots, 0, \theta_n) = e^{[{}^0\bar{\mathcal{S}}_n]\theta_n} M$

this screw motion does not chang $S_{n-1}, S_{n-2}, \ldots$

- Apply screw motion to joint $n-1$ to obtain:

$$T_{sb}(0, \ldots, 0, \theta_{n-1}, \theta_n) = e^{[{}^0\bar{\mathcal{S}}_{n-1}]\theta_{n-1}} e^{[{}^0\bar{\mathcal{S}}_n]\theta_n} M$$

- After n screw motions, the overall forward kinematics:

$$T_{sb}(\theta_1, \ldots, \theta_n) = e^{[{}^0\bar{\mathcal{S}}_1]\theta_1} e^{[{}^0\bar{\mathcal{S}}_2]\theta_2} \cdots e^{[{}^0\bar{\mathcal{S}}_n]\theta_n} M$$

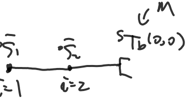
PoE

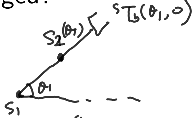
PoE: Screw Motions in Different Order (1/2)

- PoE was obtained by applying screw motions along screw axes ${}^0\bar{\mathcal{S}}_n$, ${}^0\bar{\mathcal{S}}_{n-1}$, ... What happens if the order is changed?

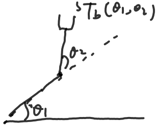
back to first example

Top view





Step 1



Step 2

- For simplicity, assume that $n = 2$, and let us apply screw motion along ${}^0\bar{\mathcal{S}}_1$ first:
  - $T_{sb}(\theta_1, 0) = e^{[{}^0\bar{\mathcal{S}}_1]\theta_1} M$
  - Now screw axis for joint 2 has been changed. The new axis ${}^0\mathcal{S}_2 = {}^0\mathcal{S}_2(\theta_1, 0) \neq {}^0\bar{\mathcal{S}}_2$. $\Longrightarrow$ $T_{sb}(\theta_1,\theta_2) \neq e^{[{}^0\bar{S}_2]\theta_2} e^{[{}^0\bar{S}_1]\theta_1} M$

$${}^sT_b(\theta_1,\theta_2) = e^{[{}^0S_2(\theta_1,0)]\theta_2} e^{[{}^0\bar{S}_1]\theta_1} M$$

Initial screw axis

$${}^0\bar{S}_2 \xrightarrow{\hat{T}_1 = e^{[{}^0\bar{S}_1]\theta_1}} \underbrace{{}^0S_2(\theta_1)}_{6\times 1} = \underbrace{[Ad_{\hat{T}_1}]}_{6\times 6} \underbrace{{}^0\bar{S}_2}_{6\times 1}$$

PoE: Screw Motions in Different Order (2/2)

- $T_{sb}(\theta_1, \theta_2) = e^{[{}^0\mathcal{S}_2]\theta_2} T_{sb}(\theta_1, 0)$

$\begin{bmatrix} [\omega'] & v' \\ 0 & 0 \end{bmatrix}$

- Recall: $[R\omega] = R[\omega]R^{-1}$

Fact: If screw axis $\underset{\begin{bmatrix} \omega' \\ v' \end{bmatrix}}{S'} = [Ad_T]\underset{\begin{bmatrix} \omega \\ v \end{bmatrix}}{S} \Longleftrightarrow \underbrace{[S']}_{4\times4} = T\underbrace{[S]}_{4\times4}T^{-1}$,

- Based on this fact:

$$\underbrace{e^{[{}^0\mathcal{S}_2(\theta_1)]\theta_2}} = e^{[[Ad_{\hat{T}_1}]{}^0\bar{\mathcal{S}}_2]\theta_2} = e^{\hat{T}_1[{}^0\bar{\mathcal{S}}_2]\hat{T}_1^{-1}\cdot\theta_2} = \hat{T}_1 e^{[{}^0\bar{\mathcal{S}}_2]\theta_2}\hat{T}_1^{-1}$$

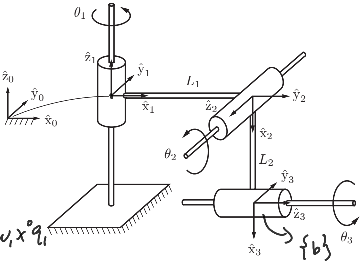
plug in $\quad {}^sT_b(\theta_1, \theta_2) = \hat{T}_1 e^{[{}^0\bar{\mathcal{S}}_2]\theta_2}\cancel{\hat{T}_1^{-1}}\cdot\cancel{e^{[{}^0\bar{\mathcal{S}}_1]\theta_1}}\cdot M = e^{[{}^0\bar{\mathcal{S}}_1]\theta_1}e^{[{}^0\bar{\mathcal{S}}_2]\theta_2}M$

$$\hat{T}_1 = e^{[{}^0\bar{\mathcal{S}}_1]\theta_1}$$

Outline

- Motivating Example
- Product of Exponential Formula Derivations
- Practice Example

PoE Example: 3R Spatial Open Chain



- Find ${}^0T_b(\theta_1, \theta_2, \theta_3)$

– step 1: Initial pose $M = \begin{bmatrix} 0 & 0 & 1 & L_1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

– step 2: ${}^0\bar{S}_1 = ({}^0\omega_1, {}^0v_1)$, ${}^0\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, ${}^0v_1 = 0 - {}^0\omega_1 \times {}^0q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

${}^0\bar{S}_2 = ({}^0\omega_2, {}^0v_2)$, ${}^0\omega_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, ${}^0v_2 = 0 - {}^0\omega_2 \times {}^0q_2 = -\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \times \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -L_1 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ -L_1 \end{bmatrix}$

${}^0\bar{S}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ * \\ * \\ * \end{bmatrix}$, ${}^0q_3 = \begin{bmatrix} L_1 \\ 0 \\ -L_2 \end{bmatrix}$, ${}^0v_3 = -\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} L_1 \\ 0 \\ -L_2 \end{bmatrix} = \begin{bmatrix} * \\ * \\ * \end{bmatrix}$

$${}^0T_b(\theta_1, \theta_2, \theta_3) = e^{[{}^0\bar{S}_1]\theta_1} \cdot e^{[{}^0\bar{S}_2]\theta_2} e^{[{}^0\bar{S}_3]\theta_3} M$$

More Discussions

-