SDM5008 Advanced Control for Robotics

Lecture 4: Instantaneous Velocity of Moving Frames

Prof. Wei Zhang

CLEAR Lab
Southern University of Science and Technology, Shenzhen, China
https://www.wzhanglab.site/

Outline

$\underline{[w] = \dot{R}R^{-1}}$ $\quad [\mathcal{V}] = \dot{T}\,T^{-1}$

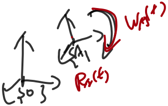
$R^{-1}\dot{R}$

- Instantaneous Velocity of Rotating Frames

- Instantaneous Velocity of Moving Frames

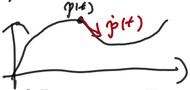
Outline

− $T_A(t) \in \mathbb{R}^{4\times 4}$: 4×4 matrix $\in SE(3)$



· $\log(T_A(t)) \to S\theta$: is $S\theta$ the velocity of $T_A(t)$?
No

e.g.



- Instantaneous Velocity of Rotating Frames

− $p(t)$: "position" vector $\Leftrightarrow T_A(t)$
$\Updownarrow$
$S\theta$

- Instantaneous Velocity of Moving Frames

− $\dot{p}(t)$: velocity vector $\Leftrightarrow$?

− $\dot{T}_A(t)$ is velocity of $T(t)$?
No

− How about $\log(\dot{T}_A(t))$ ✗ $\notin SE(3)$

Objectives

Instantaneous Velocity of Rotating Frame (1/2)

- {A} frame is rotating with orientation $R_A(t)$ and velocity $\omega_A(t)$ at time t (Note: everything is wrt {O}-frame)

$\hat{\omega}\theta \neq \omega_A(t)$

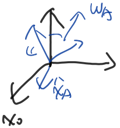
- Let $\hat{\omega}\theta = \log(\overbrace{R_A(t)}^{\in SO(3)})$ be its exp. coordinate. ("position"-vector)
  - Note: $\hat{\omega}\theta$ means $R_A(t)$ can be obtained from the reference frame (say {O}-frame) by rotating about $\hat{\omega}$ by θ degree.
  - $\hat{\omega}\theta$ only describes the current orientation of {A} relative to {O}, it does not contain info about how the frame is rotating at time t.

Instantaneous Velocity of Rotating Frame (2/2)

- What is the relation between $\omega_A(t)$ and $R_A(t)$?

$$\frac{d}{dt}R_A(t) = [\omega_A(t)]R_A(t) \Rightarrow [\omega_A(t)] = \dot{R}_A(t)R_A^{-1}(t)$$

$$R_A(t) = [\hat{x}_A(t) \quad \hat{y}_A(t) \quad \hat{z}_A(t)]$$



$\dot{\hat{x}}_A(t) = \omega_A \times \hat{x}_A$ ← coordinate free

$\dot{\hat{y}}_A(t) = \omega_A \times \hat{y}_A$, $\dot{\hat{z}}_A = \omega_A \times \hat{z}_A$

$\dot{R}_A = [\dot{\hat{x}}_A \quad \dot{\hat{y}}_A \quad \dot{\hat{z}}_A] = \omega_A \times R_A = [\omega_A] R_A$ $[{}^o\omega_A] \leftrightarrow {}^o\omega_A$

choose "o"-frame to express physics: ${}^o\dot{R}_A = [{}^o\omega_A]\,{}^oR_A \Rightarrow \boxed{[{}^o\omega_A] = {}^o\dot{R}_A\,{}^oR_A^{-1}}$

~ Additional Question: ${}^A\omega_A = ? \Rightarrow {}^A\omega_A = {}^AR_o\,{}^o\omega_A$

$\Downarrow$ $\Downarrow$

$[{}^A\omega_A]$ $[{}^A\omega_A] = [{}^AR_o\,{}^o\omega_A] = {}^AR_o[{}^o\omega_A]\,{}^AR_o^T$

$= {}^AR_o\,{}^o\dot{R}_A\,\cancel{{}^oR_A^{-1}}\,\cancel{{}^oR_A}$

Outline

$$[{}^{0}\omega_A] = {}^{0}\dot{R}_A \, {}^{0}R_A^{-1}$$

$$= {}^{0}R_A^{-1} \, {}^{0}\dot{R}_A$$

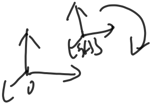
$$[{}^{A}\omega_A] = {}^{0}R_A^{-1} \, {}^{0}\dot{R}_A$$

- Instantaneous Velocity of Rotating Frames

- Instantaneous Velocity of Moving Frames

Instantaneous Velocity of Moving Frame (1/2)

- $\{A\}$ moving frame with configuration $T_A(t)$ at time t undergoes a rigid body motion with velocity $\mathcal{V}_A(t) = (\omega, v)$ (Note: everything is wrt $\{O\}$-frame)



$T_A(t) = (R_A(t), p_A(t))$

$\mathcal{V}_A$: twist of $\{A\}$

$\hat{S}\theta \not\Leftrightarrow \mathcal{V}_A = \hat{S}_A \cdot \dot{\theta}$

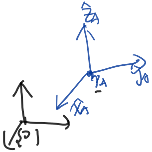
- The exponential coordinate $\hat{\mathcal{S}}\theta = \log(T_A(t))$ only indicates the current configuration of $\{A\}$, and does not tell us about how the frame is moving at time t.

- $\hat{S}\theta$: exp coordinate of T_A

- $T_A(t)$ changes with time t, we want to know $\{A\}$ spatial velocity

Instantaneous Velocity of Moving Frame (2/2)

- What is the relation between $\mathcal{V}_A(t)$ and $T_A(t)$?



$$\frac{d}{dt}T_A(t) = [\mathcal{V}_A(t)]T_A(t) \Rightarrow [\mathcal{V}_A(t)] = \dot{T}_A(t)T_A^{-1}(t)$$

$$T_A(t) = \underbrace{\begin{bmatrix} \overbrace{\hat{x}_A \quad \hat{y}_A \quad \hat{z}_A}^{R_A} & p_A \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}}_{4\times 4} = [\ \tilde{\hat{x}}_A \quad \tilde{\hat{y}}_A \quad \tilde{\hat{z}}_A \quad \tilde{p}_A\ ]$$

. Given pose $T_A(t)$ as a function of t, suppose $\{A\}$'velocity is $\mathcal{V}_A = \begin{bmatrix} \omega \\ v_r \end{bmatrix}$

any body-fixed poi

More Space

$$\dot{\tilde{\hat{X}}}_A = \begin{bmatrix} \omega \times \hat{X}_A \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} [\omega] & v_r \\ 0 & 0 \end{bmatrix}}_{[\mathcal{V}_A]}^{4\times 4} \begin{bmatrix} \hat{X}_A \\ 0 \end{bmatrix}$$

$$\dot{\tilde{\hat{X}}}_A = [\mathcal{V}_A]\, \tilde{\hat{X}}_A \quad \cdots (1)$$

similarly, $\dot{\tilde{\hat{y}}}_A = [\mathcal{V}_A]\, \tilde{\hat{y}}_A \quad \cdots (2)$ $\quad \dot{\tilde{\hat{z}}}_A = [\mathcal{V}_A]\, \tilde{\hat{z}}_A \quad \cdots (3)$

$$\dot{\tilde{p}}_A = \begin{bmatrix} \dot{p}_A \\ 0 \end{bmatrix}$$

$\dot{p}_A = v_r + \omega \times \overrightarrow{r p_A}$, choose r to be origin $\{o\}$

$$= {}^o v_o + {}^o\omega \times {}^o p_A \quad \Rightarrow \quad \begin{bmatrix} {}^o\dot{p}_A \\ 0 \end{bmatrix} = \begin{bmatrix} [{}^o\omega] & {}^o v_o \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_A \\ 1 \end{bmatrix} = [{}^o\mathcal{V}_A]\, \tilde{p}_A$$

$$\Rightarrow {}^o\dot{T}_A = [{}^o\mathcal{V}_A]\, {}^oT_A \quad \Rightarrow \quad [{}^o\mathcal{V}_A] = {}^o\dot{T}_A\, {}^oT_A^{-1} \quad \Rightarrow [{}^A\mathcal{V}_A] = {}^oT_A^{-1}\, {}^o\dot{T}_A$$