SDM5008 Advanced Control for Robotics

Lecture 3: Exponential Coordinate of Rigid Body Configuration

Prof. Wei Zhang

Southern University of Science and Technology, Shenzhen, China

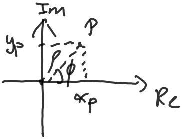
Outline

- Exponential Coordinate of $SO(3)$
- Euler Angles and Euler-Like Parameterizations
- Exponential Coordinate of SE(3)

Outline

- Exponential Coordinate of $SO(3)$
- Euler Angles and Euler-Like Parameterizations
- Exponential Coordinate of SE(3)

Towards Exponential Coordinate of $SO(3)$



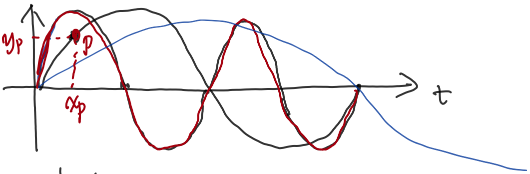
$p = (x_p, y_p)$

- Recall the polar coordinate system of the complex plane:
  - Every complex number $z = x + jy = \rho e^{j\phi}$
  - Cartesian coordinate $(x, y) \leftrightarrow$ polar coorindate (ρ, ϕ)
  - For some applications, polar coordinate is preferred due to its geometric meaning.

- Consider a set $M = \{(t, \sin(2n\pi t)) : t \in (0, 1), \underline{n = 1, 2, 3, \ldots}\}$

$M \subseteq \mathbb{R}^2$

- $p \in M$; $p = (x_p, y_p)$



- Take advantage of structure of M

coordinate of p: $(1, x_p) \longleftrightarrow (x_p, \sin(2\pi x_p))$

p': $(3, x_{p'}) \longleftrightarrow (x_{p'}, \sin(6\pi x_{p'}))$

Exponential Coordinate of $SO(3)$

$SO(3) = \{R \in \mathbb{R}^{3\times 3}: R^T R = I, \det(R) = 1\}$

- **Proposition** [Exponential Coordinate $\leftrightarrow$ SO(3)]
  - For any unit vector $[\hat{\omega}] \in so(3)$ and any $\theta \in \mathbb{R}$,

$$e^{[\hat{\omega}]\theta} \in SO(3)$$

$$R = \begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix}$$

  - For any $R \in SO(3)$, there exists $\hat{\omega} \in \mathbb{R}^3$ with $\|\hat{\omega}\| = 1$ and $\theta \in \mathbb{R}$ such that

$$R = e^{[\hat{\omega}]\theta}$$

$$\begin{array}{llcl} \text{exp:} & [\hat{\omega}]\theta \in so(3) & \xrightarrow{\exp(\cdot)} & R \in SO(3) \\ \text{log:} & R \in SO(3) & \xrightarrow{\log(\cdot)} & [\hat{\omega}]\theta \in so(3) \end{array}$$

- The vector $\hat{\omega}\theta$ is called the *exponential coordinate* for R

- The exponential coordinates are also called the canonical coordinates of the rotation group $SO(3)$

Rotation Matrix as Forward Exponential Map

- Exponential Map: By definition

$$e^{[\omega]\theta} = I + \theta[\omega] + \frac{\theta^2}{2!}[\omega]^2 + \frac{\theta^3}{3!}[\omega]^3 + \cdots \quad -- \text{①}$$

- **Rodrigues' Formula**: Given any unit vector $[\hat{\omega}] \in so(3)$, we have

$$e^{[\hat{\omega}]\theta} = I + [\hat{\omega}]\sin(\theta) + [\hat{\omega}]^2(1 - \cos(\theta))$$

Fact: if $\|\hat{w}\| = 1$, then $[\hat{w}] = -[\hat{w}]^T$, $[\hat{w}]^3 = -[\hat{w}]$

$[\hat{w}]^4 = [\hat{w}][\hat{w}]$

$= -[\hat{w}]^2$

plug into ①

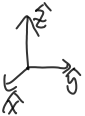
$\Rightarrow$ Rodrigues' Formula

Examples of Forward Exponential Map

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$a \times b = \underbrace{[a]}\, b$$

- Rotation matrix $R_x(\theta)$ (corresponding to $\hat{x}\theta$)



$$\hat{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow [\hat{x}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow R_x(\theta) = Rot(\hat{x};\theta)$$

$$= e^{[\hat{x}]\theta}$$

$$\Rightarrow R_x(\theta) = I + \sin\theta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + (1-\cos\theta)\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

- Rotation matrix corresponding to $(1, 0, 1)^T$

$$\underbrace{}_{\text{exp coordinate}}$$

$$\hat{\omega}\theta = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \hat{\omega} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \theta = \sqrt{2}$$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \longrightarrow R = e^{[\hat{\omega}]\theta} = \begin{bmatrix} \quad & \quad \end{bmatrix}$$

Logarithm of Rotations

- If $R = I$, then $\theta = 0$ and $\hat{\omega}$ is undefined.

- If $\mathrm{tr}(R) = -1$, then $\theta = \pi$ and set $\hat{\omega}$ equal to one of the following

$$\frac{1}{\sqrt{2(1+r_{33})}}\begin{bmatrix} r_{13} \\ r_{23} \\ 1+r_{33} \end{bmatrix}, \frac{1}{\sqrt{2(1+r_{22})}}\begin{bmatrix} r_{12} \\ 1+r_{22} \\ r_{32} \end{bmatrix}, \frac{1}{\sqrt{2(1+r_{11})}}\begin{bmatrix} 1+r_{11} \\ r_{21} \\ r_{31} \end{bmatrix}$$

- Otherwise, $\theta = \cos^{-1}\left(\frac{1}{2}(\mathrm{tr}(R) - 1)\right) \in [0, \pi)$ and $[\hat{\omega}] = \frac{1}{2\sin(\theta)}(R - R^T)$

Given any $R \in SO(3) \xrightarrow{\text{Log}}$ find $\hat{\omega}\theta$ such $e^{[\hat{\omega}]\theta} = R$

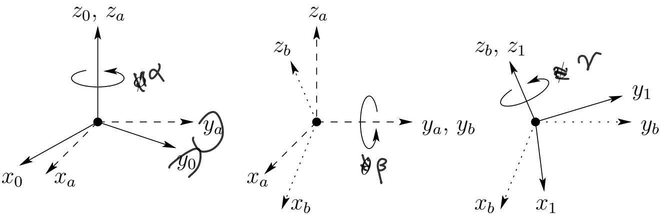
exp(·)

Outline

- Exponential Coordinate of $SO(3)$
- Euler Angles and Euler-Like Parameterizations
- Exponential Coordinate of SE(3)

Euler Angle Representation of Rotation

ZYZ
- Euler Angle
(α, β, γ)



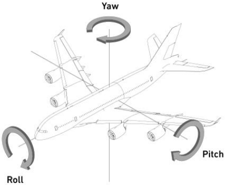
- A common method of specifying a rotation matrix is through three independent quantities called **Euler Angles**.
- Euler angle representation
  - Initially, frame $\{0\}$ coincides with frame $\{1\}$
  - Rotate $\{1\}$ about $\hat{\mathrm{z}}_0$ by an angle α, then rotate about $\hat{\mathrm{y}}_\mathrm{a}$ axis by β, and then rotate about the $\hat{\mathrm{z}}_\mathrm{b}$ axis by γ. This yields a net orientation ${}^0R_1(\alpha, \beta, \gamma)$ parameterized by the ZYZ angles $(\alpha, \beta, \gamma) \longleftrightarrow {}^0R_1$
  - ${}^0R_1(\alpha, \beta, \gamma) = R_z(\alpha)R_y(\beta)R_z(\gamma)$

$$\begin{cases} {}^0R_1(\alpha, 0, 0) = Rot(\hat{z};\alpha)\cdot I \\ {}^0R_1(\alpha, \beta, 0) = Rot(\hat{z};\alpha)Rot(\hat{y};\beta) \end{cases}$$

Other Euler-Like Parameterizations

- Other types of Euler angle parameterization can be devised using different ordered sets of rotation axes

- Common choices include:
  - ZYX Euler angles: also called *Fick angles* or yaw, pitch and roll angles
  - YZX Euler angles (Helmholtz angles)



Outline

- Exponential Coordinate of $SO(3)$
- Euler Angles and Euler-Like Parameterizations
- Exponential Coordinate of SE(3)

Exponential Map of $se(3)$: From Twist to Rigid Motion

Theorem 1 [Exponential Map of $se(3)$]: For any $\mathcal{V} = (\omega, v)$ and $\theta \in \mathbb{R}$, we have $e^{[\mathcal{V}]\theta} \in SE(3)$ $\Leftrightarrow \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$

- Case 1 ($\omega = 0$): $e^{[\mathcal{V}]\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}$
- Case 2 ($\omega \neq 0$): without loss of generality assume $\|\omega\| = 1$. Then

$$e^{[\mathcal{V}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & G(\theta)v \\ 0 & 1 \end{bmatrix}, \text{ with } G(\theta) = I\theta + (1 - \cos(\theta))[\omega] + (\theta - \sin(\theta))[\omega]^2 \quad (1)$$

For any twist $\mathcal{V}$, $\theta \longrightarrow e^{[\mathcal{V}]\theta} \in SE(\theta)$

Log of $SE(3)$: from Rigid-Body Motion to Twist

Theorem 2 [Log of $SE(3)$]: Given any $T = (R, p) \in SE(3)$, one can always find twist $\mathcal{S} = (\omega, v)$ and a scalar θ such that

$$e^{[\mathcal{S}]\theta} = T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

Matrix Logarithm Algorithm:

- If $R = I$, then set $\omega = 0$, $v = p/\|p\|$, and $\theta = \|p\|$.
- Otherwise, use matrix logarithm on $SO(3)$ to determine ω and θ from R. Then v is calculated as $v = G^{-1}(\theta)p$, where

$$G^{-1}(\theta) = \frac{1}{\theta}I - \frac{1}{2}[\omega] + \left(\frac{1}{\theta} - \frac{1}{2}\cos\frac{\theta}{2}\right)[\omega]^2$$

Exponential Coordinates of Rigid Transformation

- To sum up, screw axis $\mathcal{S} = (\omega, v)$ can be expressed as a normalized twist; its matrix representation is

$$[\mathcal{S}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in se(3)$$

- A point started at $p(0)$ at time zero, travel along screw axis $\mathcal{S}$ at unit speed for time t will end up at $\tilde{p}(t) = e^{[\mathcal{S}]t}\tilde{p}(0)$

- Given $\mathcal{S}$ we can use Theorem 1 to compute $e^{[\mathcal{S}]t} \in SE(3)$;

- Given $T \in SE(3)$, we can use Theorem 2 to find $\mathcal{S} = (\omega, v)$ and θ such that $e^{[\mathcal{S}]\theta} = T$.

- We call $\mathcal{S}\theta$ the **Exponential Coordinate** of the homogeneous transformation $T \in SE(3)$

4×4

$T = e^{[S]\theta}$ $\quad$ $S\theta \in \mathbb{R}^6$

More Space

More Space