SDM5008 Advanced Control for Robotics

Lecture 2: Operator View of Rigid-Body Transformation

Prof. Wei Zhang

Southern University of Science and Technology, Shenzhen, China

Outline

- Matrix Exponential
- Rotation Operation via Differential Equation
- Rigid-Body Operation via Differential Equation
- Rigid-Body Operation of Screw Axis

Outline

- Matrix Exponential
- Rotation Operation via Differential Equation
- Rigid-Body Operation via Differential Equation
- Rigid-Body Operation of Screw Axis

How to Solve Linear Differential Equations?

- Consider a scalar linear system: $z(t) \in \mathbb{R}$ and $a \in \mathbb{R}$ is a constant

$$\dot{z}(t) = az(t), \quad \text{with initial condition } z(0) = z_0 \tag{1}$$

- The above ODE has a unique solution: $z(t) = e^{at} \cdot z_0$

check I.C. $z(0) = z_0$,

check vector $\dot{z}(t) = a \cdot e^{at} \cdot z_0 = a \cdot z(t)$

- What about general linear systems? $\dot{x} = \overset{n \times n}{A} x + \overset{n \times m}{B} u$

$x \in \mathbb{R}^n$

$u \in \mathbb{R}^m$

What is the "Euler's Number" e?

- What is the number "e"?

· Defined as the number such that $\underline{(e^x)'} = \underline{e^x}$, $(2^x)' \neq 2^x$

$(3^x)' \neq 3^x$

$\Rightarrow \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = e^x \Rightarrow \frac{e^h - 1}{h} \xrightarrow{h\to 0} 1$

$\Rightarrow e^h \xrightarrow{h\to 0} h+1 \Rightarrow e = \lim_{h\to 0} (h+1)^{1/h}$

$\Rightarrow 2.71\ldots$

Complex Exponential

- For real variable $x \in \mathbb{R}$, Taylor series expansion for e^x around $x = 0$:

$$e^x \triangleq \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right)' = \sum_{k=0}^{\infty}\left(\frac{x^k}{k!}\right)'$$

- This can be extended to complex variables:

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

This power series is well defined for all $z \in \mathbb{C}$

let $z = j\theta$

- In particular, we have $e^{j\theta} = 1 + j\theta - \frac{\theta^2}{2} - j\frac{\theta^3}{3!} + \cdots$

- Comparing with Taylor expansions for $\cos(\theta)$ and $\sin(\theta)$ leads to the Euler's Formula

sinθ taylor expansion : $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} \cdots$

cosθ . . . . . . $\cos = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{}{6} \cdots$

$e^{j\theta} = \cos\theta + j\sin\theta$ ⟹ Euler identity

Matrix Exponential Definition

- Similar to the real and complex cases, we can define the so-called *matrix exponential*

for square matrix

$A \in \mathbb{R}^{n\times n}$

$$e^A \triangleq \sum_{k=0}^{\infty} \frac{A^k}{k!} = \underline{\left(I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots\right)} \in \mathbb{R}^{n\times n}$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$e^A = I + A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad e^A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} + \begin{bmatrix} \frac{\lambda_1^2}{2!} & 0 \\ 0 & \frac{\lambda_2^2}{2!} \end{bmatrix} + \cdots = \begin{bmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{bmatrix}$$

- This power series is well defined for any finite square matrix $A \in \mathbb{R}^{n\times n}$.

python scipy expm

Some Important Properties of Matrix Exponential

- $Ae^A = e^A A$ $\quad A \cdot \sum_{i=0}^{n} \frac{A^i}{i!} = \sum \frac{A^i}{i!} \cdot A$

  $Ae^B \neq e^B A, \quad A \neq B$

- $e^A e^B = e^{A+B}$ if $AB = BA$

- If $A = PDP^{-1}$, then $e^A = Pe^D P^{-1}$

  similarity transformation

  $\Rightarrow$ by definition:

  $$e^A = \sum_{i=0}^{\infty} \frac{A^i}{i!} = I + A + \frac{A^2}{2!} + \cdots$$

  $$= I + PDP^{-1} + \frac{PD^2P^{-1}}{2!} + \cdots$$

  $$= P \cdot (I + D + \frac{D^2}{2} + \cdots) P^{-1}$$

  $$= P e^D P^{-1}$$

  $A^0 = I$

  $A^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}$

- For every $t, \tau \in \mathbb{R}$, $e^{At} e^{A\tau} = e^{A(t+\tau)}$

- $\left(e^A\right)^{-1} = e^{-A}$

Autonomous Linear Systems

$\dot{z} = a z \Rightarrow z(t) = e^{at} \cdot z_0$

$x \in \mathbb{R}^n$ $n \times n$

$$\dot{x}(t) = Ax(t), \quad \text{with initial condition } x(0) = x_0 \tag{2}$$

- $x(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$ is constant matrix, $x_0 \in \mathbb{R}^n$ is given.
- With the definition of matrix exponential, we can show that the solution to (2) is given by

$$x(t) = e^{At} x_0$$

function of t in $\mathbb{R}^n$

proof: $\boxed{\begin{cases} \text{check I.C. (Initial condition)} \\ \text{check vector field} \end{cases}}$ $\rightarrow$ $x(0) = e^{A \cdot 0} \cdot x_0 = I \cdot x_0 = x_0$

$e^{[\ ]} = I + A + \frac{A^2}{2}$

$$\frac{d}{dt} x(t) = \frac{d}{dt}\left(e^{At} x_0\right) = \frac{d}{dt}\left(I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \cdots\right) x_0$$

$$= \left(A + \frac{A^2}{2!} 2t + \frac{A^3 \cdot 3 \cdot t^2}{3!} + \cdots\right) x_0 = \left(A + A^2 t + \frac{A^3 t^2}{2!} + \cdots\right.$$

Computation of Matrix Exponential

$$= A \cdot \left(I + At + \frac{A^2 t^2}{2!} + \cdots \right) x_0$$

$$= A e^{At} x_0$$

- Directly from definition

- For diagonalizable matrix:

- Using Padé approximation

python / by definition

Outline

All rigid body motion can be described as

$\dot{x} = Ax$

- Matrix Exponential
- Rotation Operation via Differential Equation
- Rigid-Body Operation via Differential Equation
- Rigid-Body Operation of Screw Axis

Skew Symmetric Matrices

- Recall that cross product is a special linear transformation.

- For any $\omega \in \mathbb{R}^n$, there is a matrix $[\omega] \in \mathbb{R}^{n \times n}$ such that $\omega \times p = [\omega]p$

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \leftrightarrow [\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

SO(3) : rotation matrices

- Note that $[\omega] = -[\omega]^T \leftarrow$ skew symmetric

- $[\omega]$ is called a skew-symmetric matrix representation of the vector ω

- The set of skew-symmetric matrices in: $\underbrace{so(n)}_{\text{little so(3)}} \triangleq \{S \in \mathbb{R}^{n \times n} : S^T = -S\}$

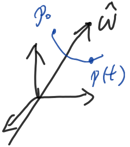
- We are interested in case $n = 2, 3$

Rotation Operation via Differential Equation

$\dot{\vec{op}}(t) = \hat{\omega} \times \vec{op}$

- Consider a point initially located at p_0 at time $t = 0$

- Rotate the point with unit angular velocity $\hat{\omega}$. Assuming the rotation axis passing through the origin, the motion is described by



$\in \mathbb{R}^3$ "A" matrix

$$\underbrace{\dot{p}(t)}_{\text{linear velocity}} = \hat{\omega} \times p(t) = [\hat{\omega}]p(t), \text{ with } p(0) = p_0 \tag{3}$$

Linear ODE: $\dot{x} = Ax$ → $x(t) = e^{At} \cdot x_0$

$p(t) = e^{[\hat{\omega}] \cdot t} \cdot p_0$

- This is a linear ODE with solution: $\boxed{p(t) = e^{[\hat{\omega}]t} p_0}$

- After $t = \theta$, the point has been rotated by θ degree. Note $p(\theta) = e^{[\hat{\omega}]\theta} p_0$

- $\boxed{\mathrm{Rot}(\hat{\omega}, \theta) \triangleq e^{[\hat{\omega}]\theta}}$ can be viewed as a rotation operator that rotates a point about $\hat{\omega}$ through θ degree

  coordinate free

Rotation Matrix as a Rotation Operator (1/3)

$R^T R = I$

theorem

- Every rotation matrix R can be written as $R = \text{Rot}(\hat{\omega}, \theta) \triangleq e^{[\hat{\omega}]\theta}$, i.e., it represents a rotation operation about $\hat{\omega}$ by θ.

is this a rotation matrix $\in SO(3)$?

Fact: any matrix of the form $e^{[\hat{\omega}]\theta}$ belongs to $SO(3)$

proof: $\left(e^{[\hat{\omega}]\theta}\right)^T \left(e^{[\hat{\omega}]\theta}\right) = I$ (V.F.T.)

$R_B = [\hat{x}_B, \hat{y}_B \ \hat{z}_B]$

- We have seen how to use R to represent frame orientation and change of coordinate between different frames. They are quite different from the operator interpretation of R.

- To apply the rotation operation, all the vectors/matrices have to be expressed in the **same reference frame** (this is clear from Eq (3))

Rotation Matrix as a Rotation Operator (2/3)

- For example, assume $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \mathrm{Rot}(\hat{\mathrm{x}}; \pi/2) = e^{[\hat{x}]\frac{\pi}{2}}$
- Consider a relation $q = Rp$:
  - **Change reference frame interpretation** : two frames $\{A\}$, $\{B\}$, one physical point a

    · R : orientation of $\{B\}$ relative to $\{A\}$, i.e. $R = {}^A R_B$

    $p = {}^B a$, $q = {}^A a$ $\quad q = Rp \Leftrightarrow {}^A a = {}^A R_B \, {}^B a$
  - **Rotation operator interpretation**:

    Have one frame $\{A\}$, two points $a \xrightarrow{\mathrm{Rot}(\cdot)} a'$, $p = {}^A a$, $q = {}^A a'$

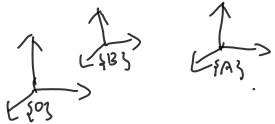
    ${}^A a' = R \; {}^A a$

    action / operator.

Rotation Matrix as a Rotation Operator (3/3)

- Consider the frame operation:
  - Change of reference frame: $\hat{R}_B = R\hat{R}_A$

orientation of A



· Have "one frame object"

· two reference frames

· Frame {A} · orientation in {o} . $^{0}R_A = [{}^{0}\hat{x}_A \ {}^{0}\hat{y}_A \ {}^{0}\hat{z}_A]$

· · · · {A} · · ·— in {B} $^{B}R_A$

$$^{0}R_A = {}^{0}R_B\,{}^{B}R_A$$

  - Rotating a frame: $R'_A = RR_A$

· two frame objects

· one reference frame {o}

$$^{0}R_{A'} = R\,{}^{0}R_A$$

Rotation Matrix Properties

- $R^T R = I$ ⇐ definition.

- $R_1 R_2 \in SO(3)$, if $R_1, R_2 \in SO(3)$.

- $\|Rp - Rq\| = \|p - q\|$ ⇐ rotation operator preserves distance

  ⇐ D V.F.T.

- $R(v \times w) = (Rv) \times (Rw)$ ⇐ rotation preserves orientation

- $R[w]R^T = [Rw]$

$[w] \in so(3)$

$R \in SO(3)$

$$[w] = \begin{bmatrix} 0 & -w_3 & w_2 \\ & 0 & -w_1 \\ & & 0 \end{bmatrix}$$

Rotation Operator in Different Frames (1/2)

- Consider two frames {A} and {B}, the actual numerical values of the operator $\mathrm{Rot}(\hat{\omega}, \theta)$ depend on both the reference frame to represent $\hat{\omega}$ and the reference frame to represent the operator itself.

- Consider a rotation axis $\hat{\omega}$ (coordinate free vector), with {A}-frame coordinate ${}^A\hat{\omega}$ and {B}-frame coordinate ${}^B\hat{\omega}$. We know

$$ {}^A\hat{\omega} = {}^A R_B \, {}^B\hat{\omega} $$

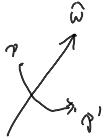
- Let ${}^B\mathrm{Rot}({}^B\hat{\omega}, \theta)$ and ${}^A\mathrm{Rot}({}^A\hat{\omega}, \theta)$ be the two rotation matrices, representing the same rotation operation $\mathrm{Rot}(\hat{\omega}, \theta)$ in frames {A} and {B}.

Rotation Operator in Different Frames (2/2)

- We have the relation:

$${}^{B}R_{A} = ({}^{A}R_{B})^{-1}$$

$${}^{A}\mathrm{Rot}({}^{A}\hat{\omega}, \theta) = {}^{A}R_{B}\, {}^{B}\mathrm{Rot}({}^{B}\hat{\omega}, \theta)\, {}^{B}R_{A}$$



Approach 1: $p \xrightarrow{\mathrm{Rot}(\hat{\omega};\theta)} p'$

use $\{A\}$ to express physics $\Rightarrow$ ${}^{A}p' = {}^{A}\mathrm{Rot}\, {}^{A}p$

$\{B\}$ frame $\Rightarrow$ ${}^{B}p' = {}^{B}\mathrm{Rot}\, {}^{B}p$

$${}^{A}R_{B}\, {}^{B}p' = {}^{A}R_{B}\, {}^{B}\mathrm{Rot}\, {}^{B}p$$

$${}^{A}p' = {}^{A}R_{B}\, {}^{B}\mathrm{Rot}\, {}^{B}R_{A}\, {}^{A}p$$

Approach 2: Recall $[Ra] = R[a]R^{T}$

A-frame rotation.

$$
\begin{aligned}
{}^{A}\mathrm{Rot} = e^{[{}^{A}\hat{\omega}]\theta} &= e^{[{}^{A}R_{B}\, {}^{B}\hat{\omega}]\theta} \\
&= e^{{}^{A}R_{B}[{}^{B}\hat{\omega}]({}^{A}R_{B})^{-1}\theta} \\
&= {}^{A}R_{B}\, e^{[{}^{B}\hat{\omega}]\theta}\, {}^{B}R_{A} \\
&= {}^{A}R_{B}\, {}^{B}\mathrm{Rot}\, {}^{B}R_{A}
\end{aligned}
$$

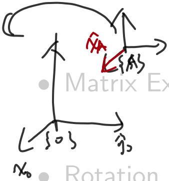
Outline

. $\{A\}$ frame $\quad Rot(\hat{x};\theta)$

$$A \xrightarrow[\{x_0\}]{\text{about}} A' \qquad \text{choose "0" frame}$$

$$^0R_{A'} = Rot(\hat{x};\theta)\, ^0R_A = e^{[^0\hat{x}_0]\frac{\pi}{2}}\, ^0R_A$$

$\{0\} \to x_0 \qquad ^0\hat{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$



- Matrix Exponential

$$A \xrightarrow[\{x_A\}]{\text{about}} A'' \Rightarrow\ ^0R_{A''} = e^{[^0\hat{x}_A]\frac{\pi}{2}}\, ^0R_A$$

- Rotation Operation via Differential Equation

$\{0\}$ frame

$$= e^{[^0R_A\, ^A\hat{x}_A]\frac{\pi}{2}}\, ^0R_A$$

- Rigid-Body Operation via Differential Equation

$$= {}^0R_A\, e^{[^A\hat{x}_A]\frac{\pi}{2}}\, \cancel{^AR_0\, ^0R_A}$$

$$= {}^0R_A\, e^{[^A\hat{x}_A]\frac{\pi}{2}} \qquad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

the same

R

- Rigid-Body Operation of Screw Axis

$$\Rightarrow\ ^0R_{A'} = R\, ^0R_A$$

$$^0R_{A''} = {}^0R_A\, R$$

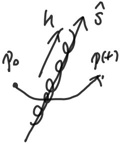
Rigid-Body Operation via Differential Equation (1/3)

- Recall: Every $R \in SO(3)$ can be viewed as the state transition matrix associated with the rotation ODE(3). It maps the initial position to the current position (after the rotation motion)
  - $p(\theta) = \mathrm{Rot}(\hat{\omega}, \theta) p_0$ viewed as a solution to $\dot{p}(t) = [\hat{\omega}]p(t)$ with $p(0) = p_0$ at $t = \theta$.
  - The above relation requires that the rotation axis passes through the origin.
- We can obtain similar ODE characterization for $T \in SE(3)$, which will lead to exponential coordinate of $SE(3)$

Rigid-Body Operation via Differential Equation (2/3)

- Recall: Theorem (Chasles): Every rigid body motion can be realized by a screw motion

- Consider a point p undergoes a screw motion with screw axis $\mathcal{S}$ and unit speed ($\dot{\theta} = 1$). Let the corresponding twist be $\mathcal{V} = \mathcal{S} = (\omega, v)$. The motion can be described by the following ODE.



$$\dot{p}(t) = \omega \times p(t) + v_r \Rightarrow \begin{bmatrix} \dot{p}(t) \\ 0 \end{bmatrix} = \overbrace{\begin{bmatrix} \overset{3\times3}{[\omega]} & v_{3\times1} \\ 0 & 0 \end{bmatrix}}^{4\times4} \begin{bmatrix} p(t) \\ 1 \end{bmatrix} \quad (4)$$

$\dot{\vec{r}}_p(t) = \omega \times \vec{r}_p(t) + v_r$

$\tilde{p}(t) = \begin{bmatrix} p(t) \\ 1 \end{bmatrix} \Rightarrow \underbrace{\dot{\tilde{p}}(t)}_{\dot{x}} = \begin{bmatrix} \dot{p}(t) \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} \ \end{bmatrix}}_{A} \underbrace{\tilde{p}(t)}_{x}$

- Solution to (4) in homogeneous coordinate is:

$$\begin{bmatrix} p(t) \\ 1 \end{bmatrix} = \exp\left(\begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} t \right) \begin{bmatrix} p(0) \\ 1 \end{bmatrix}$$

matrix representation of $\mathcal{V} = (\omega, v)$, denoted by $[\mathcal{V}]$

$\tilde{p}(t) = e^{[\ ]\cdot t}\, \tilde{p}(0)$

Rigid-Body Operation via Differential Equation (3/3)

- For any twist $\mathcal{V} = (\omega, v)$, let $[\mathcal{V}]$ be its matrix representation

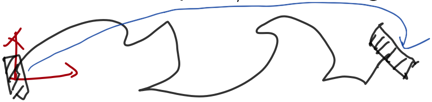
$$[\mathcal{V}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}$$

- The above definition also applies to a screw axis $\mathcal{S} = (\omega, v)$

- With this notation, the solution to (4) is $\tilde{p}(t) = e^{[\mathcal{S}]t}\tilde{p}(0)$

- Fact: $e^{[\mathcal{S}]t} \in SE(3)$ is always a valid homogeneous transformation matrix.

- Fact: Any $T \in SE(3)$ can be written as $T = e^{[\mathcal{S}]t}$, i.e., it can be viewed as an operator that moves a point/frame along the screw axis at unit speed for time t



$se(3)$

- Similar to $so(3)$, we can define $se(3)$:

$$se(3) = \{([\omega], v) : [\omega] \in so(3), v \in \mathbb{R}^3\}$$

$[\mathcal{V}]$

- $se(3)$ contains all matrix representation of twists or equivalently all twists.

- In some references, $[\mathcal{V}]$ is called a twist.

$\to [\mathcal{V}]$

- Sometimes, we may abuse notation by writing $\mathcal{V} \in se(3)$.

Homogeneous Transformation as Rigid-Body Operator

- ODE for rigid motion under $\mathcal{V} = (\omega, v)$

$$\dot{p} = v + \omega \times p \quad \Rightarrow \dot{\tilde{p}}(t) = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \tilde{p}(t) \Rightarrow \tilde{p}(t) = e^{[\mathcal{V}]t}\tilde{p}(0)$$

- Consider "unit velocity" $\mathcal{V} = \mathcal{S}$, then time t means degree

if $\mathcal{V}$ is not unit speed, $\mathcal{V} = S \cdot \dot{\theta} \Longleftrightarrow \omega = \hat{\omega}\dot{\theta}$

- $\tilde{p}' = T\tilde{p}$: "rotate" p about screw axis $\mathcal{S}$ by θ degree

$\tilde{p}' = \underbrace{e^{[S]\theta}}_{T} \tilde{p}$ $\Leftrightarrow$ use the same reference frame

choose $\{o\}$. ${}^{o}\tilde{p}' = e^{[{}^{o}S]\theta}\, {}^{o}\tilde{p}$

- TT_A: "rotate" {A}-frame about $\mathcal{S}$ by θ degree

$e^{[S]\theta} \cdot T_A \rightarrow [\tilde{x}_A \;\; \tilde{y}_A \;\; \tilde{z}_A \;\; \tilde{p}_A]$

Rigid-Body Operator in Different Frames

- Expression of T in another frame (other than {O}):

$$\underset{\text{operation in } \{O\}}{T} \quad \leftrightarrow \quad \underset{\text{operation in } \{B\}}{T_B^{-1} T T_B}$$

$${}^{B}(Rot) \;\Rightarrow\; {}^{B}R_{C}\, {}^{C}(Rot)\, {}^{C}R_{B}$$

Outline

- Matrix Exponential
- Rotation Operation via Differential Equation
- Rigid-Body Operation via Differential Equation
- Rigid-Body Operation of Screw Axis

Rigid Operation on Screw Axis

- Consider an arbitrary screw axis $\mathcal{S}$, suppose the axis has gone through a rigid transformation $T = (R, p)$ and the resulting new screw axis is $\mathcal{S}'$, then

$$\mathcal{S}' = [\mathrm{Ad}_T]\,\mathcal{S}$$

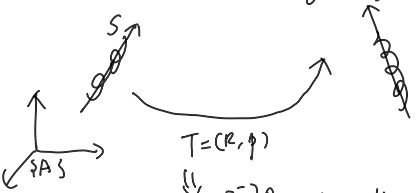
Let's work with an arbitrary frame. $\{A\}$ rigidly attached to screw axis S

proof:

$S = (w, v) \Leftrightarrow (\hat{s}, h, q)$ $\quad$ $S' = (\hat{s}', h', q')$

- Let $\{B\}$ be the frame obtained by

appl



$T = (R, p)$

$\Downarrow$

$e^{[\bar{S}]\theta}$: what's the relation between S and $\bar{S}$?

Not relation at all.

More Space