

**SDM5008 Advanced Control for Robotics**

# **Lecture 2: Operator View of Rigid-Body Transformation**

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# Outline

- Matrix Exponential
- Rotation Operation via Differential Equation
- Rigid-Body Operation via Differential Equation
- Rigid-Body Operation of Screw Axis

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# How to Solve Linear Differential Equations?

- Consider a scalar linear system:  $z(t) \in \mathbb{R}$  and  $a \in \mathbb{R}$  is a constant

$$\dot{z}(t) = az(t), \quad \text{with initial condition } z(0) = z_0 \quad (1)$$

- The above ODE has a unique solution:  $z(t) = e^{at} \cdot z_0$

check I.C.  $z(0) = z_0$ ,

check vector  $\dot{z}(t) = a \cdot e^{at} \cdot z_0 = a \cdot z(t)$

- What about general linear systems?  $\dot{x} = \overset{n \times n}{A}x + \overset{n \times m}{B}u$

$$x \in \mathbb{R}^n$$

$$u \in \mathbb{R}^m$$

# What is the "Euler's Number" $e$ ?

- What is the number "e"?

• Defined as the number such that  $\underline{(e^x)'} = \underline{e^x}$ ,  $(2^x)' \neq 2^x$

$$(3^x)' \neq 3^x$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h} = e^x \Rightarrow \frac{e^h - 1}{h} \xrightarrow{h \rightarrow 0} 1$$

$$\Rightarrow e \xrightarrow{h \rightarrow 0} h+1 \Rightarrow e = \lim_{h \rightarrow 0} (h+1)^{1/h} \Rightarrow 2.71\dots$$

# Complex Exponential

- For real variable  $x \in \mathbb{R}$ , Taylor series expansion for  $e^x$  around  $x = 0$ :

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

- This can be extended to complex variables:

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

This power series is well defined for all  $z \in \mathbb{C}$

- In particular, we have  $e^{j\theta} = 1 + j\theta - \frac{\theta^2}{2} - j\frac{\theta^3}{3!} + \dots$
- Comparing with Taylor expansions for  $\cos(\theta)$  and  $\sin(\theta)$  leads to the Euler's Formula  $\sin \theta$  Taylor expansion:  $\sin \theta$

$$\left( \sum_{k=0}^{\infty} \frac{x^k}{k!} \right)' = \sum_{k=0}^{\infty} \left( \frac{x^k}{k!} \right)'$$

(let  $z = j\theta$ )

# Matrix Exponential Definition

- Similar to the real and complex cases, we can define the so-called *matrix exponential*

$$e^A \triangleq \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$

- This power series is well defined for any finite square matrix  $A \in \mathbb{R}^{n \times n}$ .

# Some Important Properties of Matrix Exponential

- $Ae^A = e^A A$
- $e^A e^B = e^{A+B}$  if  $AB = BA$
- If  $A = PDP^{-1}$ , then  $e^A = Pe^D P^{-1}$
- For every  $t, \tau \in \mathbb{R}$ ,  $e^{At} e^{A\tau} = e^{A(t+\tau)}$
- $(e^A)^{-1} = e^{-A}$



# Autonomous Linear Systems

$$\dot{x}(t) = Ax(t), \quad \text{with initial condition } x(0) = x_0 \quad (2)$$

- $x(t) \in \mathbb{R}^n$ ,  $A \in \mathbb{R}^{n \times n}$  is constant matrix,  $x_0 \in \mathbb{R}^n$  is given.
- With the definition of matrix exponential, we can show that the solution to (2) is given by

$$x(t) = e^{At}x_0$$

# Computation of Matrix Exponential

- Directly from definition
- For diagonalizable matrix:
- Using Padé approximation

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# Skew Symmetric Matrices

- Recall that cross product is a special linear transformation.
- For any  $\omega \in \mathbb{R}^n$ , there is a matrix  $[\omega] \in \mathbb{R}^{n \times n}$  such that  $\omega \times p = [\omega]p$

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \leftrightarrow [\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

- Note that  $[\omega] = -[\omega]^T \leftarrow$  skew symmetric
- $[\omega]$  is called a skew-symmetric matrix representation of the vector  $\omega$
- The set of skew-symmetric matrices in:  $so(n) \triangleq \{S \in \mathbb{R}^{n \times n} : S^T = -S\}$
- We are interested in case  $n = 2, 3$

## Rotation Operation via Differential Equation

- Consider a point initially located at  $p_0$  at time  $t = 0$
- Rotate the point with unit angular velocity  $\hat{\omega}$ . Assuming the rotation axis passing through the origin, the motion is described by

$$\dot{p}(t) = \hat{\omega} \times p(t) = [\hat{\omega}]p(t), \text{ with } p(0) = p_0 \quad (3)$$

- This is a linear ODE with solution:  $p(t) = e^{[\hat{\omega}]t}p_0$
- After  $t = \theta$ , the point has been rotated by  $\theta$  degree. Note  $p(\theta) = e^{[\hat{\omega}]\theta}p_0$
- $\text{Rot}(\hat{\omega}, \theta) \triangleq e^{[\hat{\omega}]\theta}$  can be viewed as a rotation operator that rotates a point about  $\hat{\omega}$  through  $\theta$  degree

## Rotation Matrix as a Rotation Operator (1/3)

- Every rotation matrix  $R$  can be written as  $R = \text{Rot}(\hat{\omega}, \theta) \triangleq e^{[\hat{\omega}]\theta}$ , i.e., it represents a rotation operation about  $\hat{\omega}$  by  $\theta$ .
  
- We have seen how to use  $R$  to represent frame orientation and change of coordinate between different frames. They are quite different from the operator interpretation of  $R$ .
  
- To apply the rotation operation, all the vectors/matrices have to be expressed in the **same reference frame** (this is clear from Eq (3))

## Rotation Matrix as a Rotation Operator (2/3)

- For example, assume  $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \text{Rot}(\hat{x}; \pi/2)$
- Consider a relation  $q = Rp$ :
  - **Change reference frame interpretation :**
  
  
  
  
  
  
  
  
  
  
  - **Rotation operator interpretation:**

## Rotation Matrix as a Rotation Operator (3/3)

- Consider the frame operation:
  - Change of reference frame:  $R_B = RR_A$

- Rotating a frame:  $R'_A = RR_A$



# Rotation Matrix Properties

- $R^T R = I$
- $R_1 R_2 \in SO(3)$ , if  $R_1, R_2 \in SO(3)$
- $\|Rp - Rq\| = \|p - q\|$
- $R(v \times w) = (Rv) \times (Rw)$
- $R[w]R^T = [Rw]$

## Rotation Operator in Different Frames (1/2)

- Consider two frames  $\{A\}$  and  $\{B\}$ , the actual numerical values of the operator  $\text{Rot}(\hat{\omega}, \theta)$  depend on both the reference frame to represent  $\hat{\omega}$  and the reference frame to represent the operator itself.
- Consider a rotation axis  $\hat{\omega}$  (coordinate free vector), with  $\{A\}$ -frame coordinate  ${}^A\hat{\omega}$  and  $\{B\}$ -frame coordinate  ${}^B\hat{\omega}$ . We know

$${}^A\hat{\omega} = {}^A R_B {}^B\hat{\omega}$$

- Let  ${}^B\text{Rot}({}^B\hat{\omega}, \theta)$  and  ${}^A\text{Rot}({}^A\hat{\omega}, \theta)$  be the two rotation matrices, representing the same rotation operation  $\text{Rot}(\hat{\omega}, \theta)$  in frames  $\{A\}$  and  $\{B\}$ .

## Rotation Operator in Different Frames (2/2)

- We have the relation:

$${}^A\text{Rot}({}^A\hat{\omega}, \theta) = {}^A R_B {}^B\text{Rot}({}^B\hat{\omega}, \theta) {}^B R_A$$

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## Rigid-Body Operation via Differential Equation (1/3)

- Recall: Every  $R \in SO(3)$  can be viewed as the state transition matrix associated with the rotation ODE(3). It maps the initial position to the current position (after the rotation motion)
  - $p(\theta) = \text{Rot}(\hat{\omega}, \theta)p_0$  viewed as a solution to  $\dot{p}(t) = [\hat{\omega}]p(t)$  with  $p(0) = p_0$  at  $t = \theta$ .
  - The above relation requires that the rotation axis passes through the origin.
- We can obtain similar ODE characterization for  $T \in SE(3)$ , which will lead to exponential coordinate of  $SE(3)$

## Rigid-Body Operation via Differential Equation (2/3)

- Recall: Theorem (Chasles): Every rigid body motion can be realized by a screw motion
- Consider a point  $p$  undergoes a screw motion with screw axis  $\mathcal{S}$  and unit speed ( $\dot{\theta} = 1$ ). Let the corresponding twist be  $\mathcal{V} = \mathcal{S} = (\omega, v)$ . The motion can be described by the following ODE.

$$\dot{p}(t) = \omega \times p(t) + v \quad \Rightarrow \quad \begin{bmatrix} \dot{p}(t) \\ 0 \end{bmatrix} = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ 1 \end{bmatrix} \quad (4)$$

- Solution to (4) in homogeneous coordinate is:

$$\begin{bmatrix} p(t) \\ 1 \end{bmatrix} = \exp \left( \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} t \right) \begin{bmatrix} p(0) \\ 1 \end{bmatrix}$$

## Rigid-Body Operation via Differential Equation (3/3)

- For any twist  $\mathcal{V} = (\omega, v)$ , let  $[\mathcal{V}]$  be its matrix representation

$$[\mathcal{V}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}$$

- The above definition also applies to a screw axis  $\mathcal{S} = (\omega, v)$
- With this notation, the solution to (4) is  $\tilde{p}(t) = e^{[\mathcal{S}]t}\tilde{p}(0)$
- Fact:  $e^{[\mathcal{S}]t} \in SE(3)$  is always a valid homogeneous transformation matrix.
- Fact: Any  $T \in SE(3)$  can be written as  $T = e^{[\mathcal{S}]t}$ , i.e., it can be viewed as an operator that moves a point/frame along the screw axis at unit speed for time  $t$

## $se(3)$

- Similar to  $so(3)$ , we can define  $se(3)$ :

$$se(3) = \{([\omega], v) : [\omega] \in so(3), v \in \mathbb{R}^3\}$$

- $se(3)$  contains all matrix representation of twists or equivalently all twists.
- In some references,  $[\mathcal{V}]$  is called a twist.
- Sometimes, we may abuse notation by writing  $\mathcal{V} \in se(3)$ .



# Homogeneous Transformation as Rigid-Body Operator

- ODE for rigid motion under  $\mathcal{V} = (\omega, v)$

$$\dot{p} = v + \omega \times p \quad \Rightarrow \quad \dot{\tilde{p}}(t) = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \tilde{p}(t) \Rightarrow \tilde{p}(t) = e^{[\mathcal{V}]t} \tilde{p}(0)$$

- Consider “unit velocity”  $\mathcal{V} = \mathcal{S}$ , then time  $t$  means degree
- $\tilde{p}' = T\tilde{p}$ : “rotate”  $p$  about screw axis  $\mathcal{S}$  by  $\theta$  degree
- $TT_A$ : “rotate”  $\{A\}$ -frame about  $\mathcal{S}$  by  $\theta$  degree

## Rigid-Body Operator in Different Frames

- Expression of  $T$  in another frame (other than  $\{O\}$ ):

$$\begin{array}{ccc} T & \leftrightarrow & T_B^{-1} T T_B \\ \text{operation in } \{O\} & & \text{operation in } \{B\} \end{array}$$

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## Rigid Operation on Screw Axis

- Consider an arbitrary screw axis  $\mathcal{S}$ , suppose the axis has gone through a rigid transformation  $T = (R, p)$  and the resulting new screw axis is  $\mathcal{S}'$ , then

$$\mathcal{S}' = [\text{Ad}_T] \mathcal{S}$$

proof:

# More Space