SDM5008 Advanced Control for Robotics

# Lecture 1: Rigid Body Configuration and Velocity

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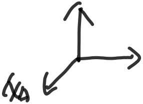
# Outline

- Rigid Body Configuration
- Rigid Body Velocity (Twist)
- Geometric Aspect of Twist: Screw Motion

# Free Vector

- **Free Vector**: geometric quantity with length and direction

- Given a reference frame, $v$ can be moved to a position such that the base of the arrow is at the origin without changing the orientation. Then the vector $v$ can be represented by its coordinates $v$ in the reference frame.



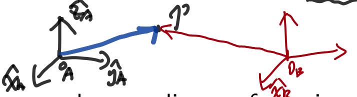
- $v$ denotes the physical quantity while ${}^{A}v$ denote its coordinate wrt frame $\{A\}$.

# Point

- **Point**: $p$ denotes a point in the physical space

$\cdot\, p$

- A point $p$ can be represented by a vector from frame origin to $p$




$${}^A p \triangleq {}^A(\overrightarrow{O_A p}), \quad {}^B p = {}^B(\overrightarrow{O_B p})$$

- ${}^A p$ denotes the coordinate of a point $p$ wrt frame $\{A\}$

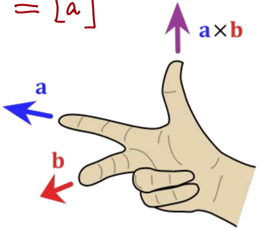
$${}^B(\overrightarrow{O_A p})$$

- When left-superscript is not present, it means the physical vector itself or the coordinate of the vector for which the reference frame is clear from the context.

# Cross Product

- **Cross product** or **vector product** of $a \in \mathbb{R}^3, b \in \mathbb{R}^3$ is defined as

$$a \times b = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -a_3 & -a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}}_{=[a]} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad (1)$$




**Properties:**

- $\|a \times b\| = \|a\|\|b\| \sin(\theta)$
- $a \times b = -b \times a$
- $a \times a = 0$

# Skew symmetric representation

- It can be directly verified from definition that $a \times b = [a]b$, where

$A = A^T$

$A = -A^T$

$$[a] \triangleq \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \quad (2)$$

- $a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \leftrightarrow [a]$

- $[a] = -[a]^T$ (called skew symmetric)

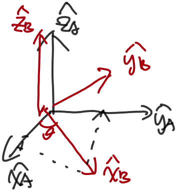
- $[a][b] - [b][a] = [a \times b]$ (Jacobi's identity)

# Rotation Matrix

- **Frame**: 3 coordinate vectors (unit length) $\hat{x}, \hat{y}, \hat{z}$, and an origin
  - $\hat{\mathrm{x}}, \hat{\mathrm{y}}, \hat{\mathrm{z}}$ mutually orthogonal . $\hat{x}^T\hat{y}=0$ , $\hat{x}^T\hat{z}=0$ $\hat{y}^T\hat{z}=0$
  - $\hat{\mathrm{x}} \times \hat{\mathrm{y}} = \hat{\mathrm{z}} \Leftarrow$ right hand rule

- **Rotation Matrix**: specifies orientation of one frame relative to another

$$ {}^{A}R_{B} \triangleq \left[ \; {}^{A}\hat{x}_{B} \quad {}^{A}\hat{y}_{B} \quad {}^{A}\hat{z}_{B} \; \right] $$




$$ {}^{A}R_{B} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{A}R_{B}^{T} = \begin{bmatrix} {}^{A}\hat{x}_{B}^{T} \\ {}^{A}\hat{y}_{B}^{T} \\ {}^{A}\hat{z}_{B}^{T} \end{bmatrix} \left[ {}^{A}\hat{x}_{B} \cdots \right] $$

- A valid rotation matrix $R$ satisfies: (i) $R^T R = I$; (ii) $\det(R) = 1$

$$ \det({}^{A}R_{B}) $$
$$ = {}^{A}\hat{x}_{B}^{T} \left( {}^{A}\hat{y}_{B} \times {}^{A}\hat{z}_{B} \right) = 1 $$

# Special Orthogonal Group

$SO(3)$, $SO(2)$

- **Special Orthogonal Group**: Space of Rotation Matrices in $\mathbb{R}^n$ is defined as

$$SO(n) = \{R \in \mathbb{R}^{n\times n} : R^T R = I, \det(R) = 1\}$$

- $SO(n)$ is a *group*. We are primarily interested in $SO(3)$ and $SO(2)$, rotation groups of $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively.

- **Group** is a set $G$, together with an operation $\bullet$, satisfying the following group axioms:
  - **Closure:** $a \in G, b \in G \Rightarrow a \bullet b \in G$
  - **Associativity:** $(a \bullet b) \bullet c = a \bullet (b \bullet c)$, $\forall a, b, c \in G$
  - **Identity element:** $\exists e \in G$ such that $e \bullet a = a$, for all $a \in G$.
  - **Inverse element:** For each $a \in G$, there is a $b \in G$ such that $a \bullet b = b \bullet a = e$, where $e$ is the identity element.

clock arithmetic

$\{ \{0, 1, 2, 3\}, \text{"+"} \}$


$\begin{cases} 0+2=2 \\ 2+2=0 \end{cases}$

geometry study of "symmetry".

# Use of Rotation Matrix (1/2)

- Representing an orientation ${}^A R_B$ : directly from definition.

  orientation of $\{B\}$ relative to $\{A\}$

- Changing the reference frame ${}^A R_B$:

Given vector $v$, its coordinate in $\{A\}$, $\{B\}$ are ${}^A v$, ${}^B v$

$${}^A v = {}^A R_B \, {}^B v$$

proof. "coordinate-free"

- we have only one vector $v$, ${}^A v = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$, ${}^B v = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$

"physics" $v = \alpha_1 \hat{x}_A + \alpha_2 \hat{y}_A + \alpha_3 \hat{z}_A$

$= \beta_1 \hat{x}_B + \beta_2 \hat{y}_B + \beta_3 \hat{z}_B$

$\Rightarrow \alpha_1 \hat{x}_A + \alpha_2 \hat{y}_A + \alpha_3 \hat{z}_A = \beta_1 \hat{x}_B + \beta_2 \hat{y}_B + \beta_3 \hat{z}_B$ .. "physics"

$\Rightarrow$ express "physics" in $\{A\}$

# Use of Rotation Matrix (2/2)

$$\underbrace{\alpha_1 \, {}^A\hat{x}_A + \alpha_2 \, {}^A\hat{y}_A + \alpha_3 \, {}^A\hat{z}_A} = \beta_1 \, {}^A\hat{x}_B + \beta_2 \, {}^A\hat{y}_B + \beta_3 \, {}^A\hat{z}_B$$

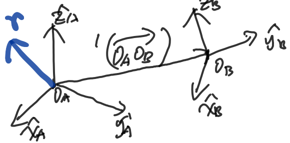
$$\underbrace{\begin{bmatrix} {}^A\hat{x}_A & {}^A\hat{y}_A & {}^A\hat{z}_A \end{bmatrix}}_{I} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}}_{{}^A v} = \underbrace{\begin{bmatrix} {}^A\hat{x}_B & {}^A\hat{y}_B & {}^A\hat{z}_B \end{bmatrix}}_{\triangleq {}^A R_B} \underbrace{\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}}_{{}^B v}$$

$${}^A v = {}^A R_B \, {}^B v$$

- Rotating a vector or a frame $\mathrm{Rot}(\hat{\omega}, \theta)$: will be discussed in next lecture.

# Rigid Body Configuration (pose)

- Given two coordinate frames {A} and {B}, the configuration of B relative to A is determined by
  - ${}^{A}R_{B}$ and ${}^{A}o_{B}$




$${}^{A}R_{B} = [{}^{A}\hat{x}_{B} \cdots]$$

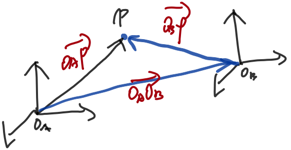
$${}^{A}o_{B} = {}^{A}(\overrightarrow{O_A O_B})$$

$$\begin{pmatrix} R_B \\ o_B \end{pmatrix}$$

- For a (free) vector $r$, its coordinates ${}^{A}r$ and ${}^{B}r$ are related by:

$${}^{A}r = {}^{A}R_{B}\,{}^{B}r \ ✓$$

- For a point $p$, its coordinates ${}^{A}p$ and ${}^{B}p$ are related by:




$${}^{A}p = {}^{A}(\overrightarrow{O_A P}), \quad {}^{B}p = {}^{B}(\overrightarrow{O_B P})$$

$${}^{B}p = {}^{B}(\overrightarrow{O_B P})$$

"physics": $\overrightarrow{O_A P} = \overrightarrow{O_A O_B} + \overrightarrow{O_B P}$

chose {A} $\Longrightarrow$ to express physics

$${}^{A}(\overrightarrow{O_A P}) = {}^{A}(\overrightarrow{O_A O_B}) + {}^{A}(\overrightarrow{O_B P})$$

$${}^{A}p = {}^{A}o_{B} + {}^{A}R_{B}\,{}^{B}p \quad \cdots (1)$$

# Homogeneous Transformation Matrix

$y = ax$

$y = ax + b$

$$\begin{bmatrix} y \\ 1 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$$

- **Homogeneous Transformation Matrix**: ${}^{A}T_{B}$

$$\underbrace{{}^{A}p}_{3\times 1} = \underbrace{{}^{A}O_{B}}_{3\times 1} + \underbrace{{}^{A}R_{B}}_{3\times 3}\,\underbrace{{}^{B}p}_{3\times 1} \Rightarrow \underbrace{\begin{bmatrix} {}^{A}p \\ 1 \end{bmatrix}}_{{}^{A}\tilde{p}} = \underbrace{\left[\begin{array}{c:c} {}^{A}R_{B} & {}^{A}O_{B} \\ \hdashline \underbrace{0}_{1\times 3} & \underbrace{1}_{1\times 1} \end{array}\right]}_{4\times 4} \underbrace{\begin{bmatrix} {}^{B}p \\ 1 \end{bmatrix}}_{{}^{B}\tilde{p}}$$

$${}^{A}T_{B} = \left[\begin{array}{c:c} {}^{A}R_{B} & {}^{A}O_{B} \\ \hdashline 0 & 1 \end{array}\right] \quad 4\times 4$$

$T = (R, p)$ : pose of a frame $\{B\}$ relative to $\{A\}$

- Homogeneous coordinates:

Given a point $p \in \mathbb{R}^3$, its homogeneous coordinate is defined as

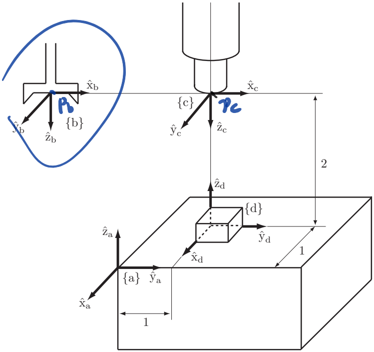
$$\tilde{p} = \begin{bmatrix} p \\ 1 \end{bmatrix} \in \mathbb{R}^4 \qquad {}^{A}\tilde{p} = {}^{A}T_{B}\,{}^{B}\tilde{p}$$

Given a vector $v \in \mathbb{R}^3$, its homo- coordinate is $\tilde{v} = \begin{bmatrix} v \\ 0 \end{bmatrix}$

$$v = p_1 - p_2 \qquad \tilde{v} = \tilde{p}_1 - \tilde{p}_3 = \begin{bmatrix} v \\ 0 \end{bmatrix} \qquad {}^{A}\tilde{v} = {}^{A}T_{B}\,{}^{B}\tilde{v}$$

# Example of Homogeneous Transformation Matrix

Fixed frame {a}; end effector frame {b}, the camera frame {c}, and the workpiece frame {d}. Suppose $\|p_c - p_b\| = 4$




1. Camera "location"? ${}^aT_c = ({}^aR_c, {}^ap_c)$

$${}^aR_c = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad {}^ap_c = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$${}^aT_c = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. end-effector frame ${}^aT_b$.

$${}^cT_b = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

${}^a\hat{x}_b$

$${}^aT_b = {}^aT_c\, {}^cT_b$$

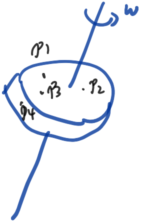
$${}^aT_b = \begin{bmatrix} {}^a\hat{x}_b & {}^a\hat{y}_b & {}^a\hat{z}_b & {}^ao_b \end{bmatrix}$$

# Outline

- Rigid Body Configuration
- Rigid Body Velocity (Twist) spatial velocity
- Geometric Aspect of Twist: Screw Motion

# Rigid Body Velocity (1/3)

- Consider a rigid body in motion. The body has infinitely many points $\{p_i\}$ with different velocities $\{v_{p_i}\}$



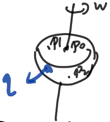

parameter common to all points on the body

$$v_{p_1} = g(p_1, \text{parameter})$$
$$v_{p_2} = g(p_2, \text{parameter})$$
$$v_{p_3} = \quad \vdots \qquad \vdots$$
$$\vdots$$

  - All these velocities $v_{p_i}$'s are not independent
  - They can be expressed by the same set of parameter
  - Rigid body velocity (i.e. spatial velocity, twist) is one such parameterization

# Rigid Body Velocity (2/3)

- Pure rotation case




Assume $p_0$ on the rotation axis

$$v_{p_i} = \omega \times \overrightarrow{p_0 p_i} = g(p_i, \omega) \quad , \quad \underline{v_{p_0} = 0}$$

- General motion

1°. Again, assume $p_0$ on rotation axis / body fixed.

$$v_{p_i} = \dot{p}_i(t) = \left( p_0(t) + \overrightarrow{p_0 p_i}(t) \right)' = v_{p_0} + \omega \times \overrightarrow{p_0 p_i} = g(p_i, para)$$

$para = (v_{p_0}, \omega)$

$\underline{v_{p_i} = v_{p_0} + \omega \times \overrightarrow{p_0 p_i}}$ ①

In this case, $\underline{p_0}$ is a reference point we use to express velocities of all the other points

2°. What is ref point NOT on rotation axis

e.g. consider arbitrary body-fixed $q$, with velocity $v_q$

– we still have the SAME expression. $\boxed{v_{p_i} = v_q + \omega \times \overrightarrow{q p_i}}$ ... ②

# Rigid Body Velocity (3/3)

- why? use $p_o$ as ~~a~~ intermediate variable

q is body-fixed $\Rightarrow$ by ① $\Rightarrow$ $v_q = v_{p_o} + \omega \times \overrightarrow{p_o q}$

$\longrightarrow$ $v_{p_i} = v_{p_o} + \omega \times \overrightarrow{p_o p_i}$

$\Rightarrow$ $v_{p_i} = v_q - \omega \times \overrightarrow{p_o q} + \omega \times \overrightarrow{p_o p_i}$

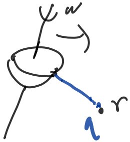
$\left( \overrightarrow{p_o p_i} - \overrightarrow{p_o q} = \overrightarrow{q p_i} \right)$

$= v_q + \omega \times \overrightarrow{q p_i}$

3 what if the reference point "r" is NOT body-fixed

(e.g. r is stationary, or move in another way)

- let q be the body-fixed point "currently coincide with r"



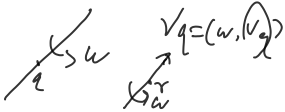

i.e. $q(t) = r$ at time $t$ ($q(t_1)$ may not equal $r$ at $t_1 \neq t$)

By ② $v_{p_i} = v_{q(t)} + \omega \times \overrightarrow{q(t) p_i(t)}$

If we define "$v_r$" $\triangleq v_{q(t)}$ then $v_{p_i} =$ "$v_r$" $+ \omega \times \overrightarrow{r p_i}$

# Rigid Body Velocity: Spatial Velocity (Twist)

- How to represent a rigid body velocity?
  - Pick an arbitrary point $r$ (reference point), which may or may not be body-fixed
  - Define $v_r$ as the velocity of **the body-fixed point currently coincides with** $r$
  - For any body-fixed point $p$ on the body: $v_p = \text{``}v_r\text{''} + \omega \times (\overrightarrow{rp})$ ⇐ coordinate free
- **Spatial Velocity (Twist)**: $\mathcal{V}_r = (\omega, v_r)$
- Twist is a "physical" quantity (just like linear or angular velocity)
  - It can be represented in any frame for any chosen reference point $r$
- A rigid body B with $\mathcal{V}_r = (\omega, v_r)$ can be "thought of" as translating at $v_r$ while rotating with angular velocity $\omega$ about an axis passing through $r$
  - This is just one way to interpret the motion.



# Spatial Velocity Representation in a Reference Frame

- Given frame $\{A\}$ and a spatial velocity $\mathcal{V}$ of body $B$

$T_A = \{R_A, o_A\}$

convention

- Choose $o_A$ (the origin of $\{A\}$) as the reference point to represent the rigid body velocity

$$^A\mathcal{V}_{o_A} = \begin{bmatrix} {}^A\omega \\ {}^A v_{o_A} \end{bmatrix}$$

- Coordinates of $\mathcal{V}$ in $\{A\}$:

$$^A\mathcal{V}_{o_A} = ({}^A\omega, {}^A v_{o_A})$$

- By default, we assume the origin of the frame is used as the reference point: $^A\mathcal{V} = {}^A\mathcal{V}_{o_A}$

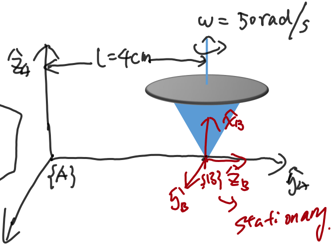
Let $p$ be a body-fixed point of body $B$

"physics": $v_p = v_{o_A} + \omega \times (\overrightarrow{o_A p}) \xRightarrow[\text{to express physics}]{\text{using } \{A\}}$

$$\begin{aligned} {}^A v_p &= {}^A v_{o_A} + {}^A\omega \times {}^A(\overrightarrow{o_A p}) \\ &= {}^A v_{o_A} + {}^A\omega \times {}^A p \end{aligned}$$

# Example of Twist I

- Example I: what's the twist of the spinning top?




Choose {A}-frame:

$$ {}^{A}\mathcal{V}_{top} = ({}^{A}\omega, {}^{A}v_{o_A}) = \begin{bmatrix} {}^{A}\omega \\ \hdashline {}^{A}v_{o_A} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 50 \\ \hdashline 2 \\ 0 \\ 0 \end{bmatrix} $$

$$ {}^{A}\omega = \begin{bmatrix} 0 \\ 0 \\ 50 \end{bmatrix} $$

$$ {}^{A}v_{o_A} = \begin{bmatrix} 50 \times 0.04 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} $$

coordinate-free

$$ {}^{A}v_{o_A} = {}^{A}v_B + {}^{A}\omega \times {}^{A}\vec{o}_{BA} $$

$$ = 0 + \begin{bmatrix} 0 \\ 0 \\ 50 \end{bmatrix} \times \begin{bmatrix} 0 \\ -0.04 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} $$

choose frame {B}

$$ {}^{B}\mathcal{V}_{top} = \begin{bmatrix} {}^{B}\omega \\ {}^{B}v_{o_B} \end{bmatrix} = \begin{bmatrix} 50 \\ 0 \\ 0 \\ \hdashline 0 \\ 0 \\ 0 \end{bmatrix} $$

revisit using ${}^{A}X_B$

$$ {}^{A}\mathcal{V}_{top} = {}^{A}X_B \, {}^{B}\mathcal{V}_{top} $$

$$ = \left[ \begin{array}{c|c} & \\ \hline & \end{array} \right] \begin{bmatrix} 0 \\ 0 \\ 50 \\ 0 \\ 0 \\ 0 \end{bmatrix} $$

$V, F, T$

$$ {}^{A}X_B = [Ad_T] = \left[ \begin{array}{ccc|ccc} 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & 0 & \\ 1 & 0 & 0 & & & \\ \hline & & & 0 & 1 & 0 \\ & \times & & 0 & 0 & 1 \\ & & & 1 & 0 & 0 \end{array} \right] $$

$$ = \left[ \begin{array}{c|c} R & 0 \\ \hline [p]R & R \end{array} \right] $$

$$ R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad p = \begin{bmatrix} 0 \\ 0.04 \\ 0 \end{bmatrix}, \quad [p] = \begin{bmatrix} 0 & \times & \times \\ - & 0 & \times \\ \times & \times & 0 \end{bmatrix} $$

# Example of Twist II

- Example II: what's the ${}^s\mathcal{V}_{car}$, ${}^b\mathcal{V}_{car}$

car rotates about r with angular velocity w

$${}^s\mathcal{V}_{car} = \begin{bmatrix} {}^s\omega \\ {}^s v_{O_s} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ -2 \\ -4 \\ 0 \end{bmatrix}, \qquad {}^s w = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \quad {}^s v_{O_s} = \begin{bmatrix} -2 \\ -4 \\ 0 \end{bmatrix}$$

$${}^s v_{O_s} = {}^s v_r + {}^s w \times {}^s\overrightarrow{rO_s}$$

$$= 0 + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \times (-{}^s r)$$

$$= \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \times \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ 0 \end{bmatrix}$$

$${}^b\mathcal{V}_{car} = \begin{bmatrix} {}^b\omega \\ {}^b v_{O_b} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 2.8 \\ 4 \\ 0 \end{bmatrix}$$

$${}^b v_{O_b} = {}^b v_r + {}^b w \times {}^b\overrightarrow{(rO_b)}$$

$$= 0 + \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \times \begin{bmatrix} -2 \\ 1.4 \\ 0 \end{bmatrix}$$

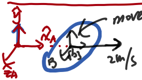
$$= \begin{bmatrix} 2.8 \\ 4 \\ 0 \end{bmatrix}$$

$${}^b v_r \stackrel{?}{=} 0$$

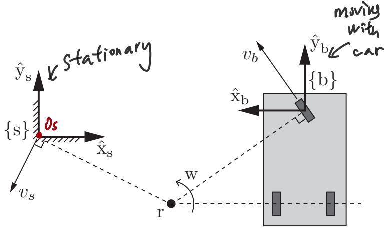
$${}^b(v_{r/s})$$

$v_r$ : "r" velocity relative to {s}

inertia




$${}^B v_b = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \qquad {}^A v_b = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$



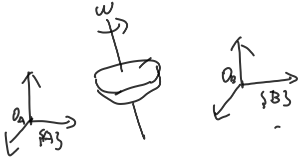

${}^s r_s = (2, -1, 0)$, ${}^b r_b = (2, -1.4, 0)$, w=2 rad/s

# Change Reference Frame for Twist (1/2)

- Given a twist $\mathcal{V}$, let ${}^A\mathcal{V}$ and ${}^B\mathcal{V}$ be their coordinates in frames {A} and {B}

$$ {}^A\mathcal{V} = \begin{bmatrix} {}^A\omega \\ {}^A v_A \end{bmatrix}, \qquad {}^B\mathcal{V} = \begin{bmatrix} {}^B\omega \\ {}^B v_B \end{bmatrix} \quad {}^B v_{O_B} \quad {}^B v_B $$

- They are related by ${}^A\mathcal{V} = {}^A X_B \, {}^B\mathcal{V}$




① ${}^A\omega = {}^A R_B \, {}^B\omega$ ....

② "coordinate-free"

$v_{O_A}$ : velocity of body-fixed pt currently coincides with $O_A$

$v_{O_B}$ : . . . . . . . . . . . $O_B$

$v_{O_A} = v_{O_B} + \omega \times \overrightarrow{O_B O_A}$ $\quad$ choose {A} to express physics.

$$ {}^A v_{O_A} = {}^A v_{O_B} + {}^A\omega \times {}^A(\overrightarrow{O_B O_A}) $$

$$ = {}^A R_B \, {}^B v_{O_B} + {}^A R_B \, {}^B\omega \times (-{}^A O_B) $$

$$ = {}^A R_B \, {}^B v_{O_B} + {}^A O_B \times {}^A R_B \, {}^B\omega $$

$$ = {}^A R_B \, {}^B v_{O_B} + [{}^A O_B] \, {}^A R_B \, {}^B\omega = $$

$a \times b = -b \times a$

$a \times b = [a] b$

# Change Reference Frame for Twist (2/2)

$\Downarrow$

$\Rightarrow {}^Av_{o_A} = \left[ [{}^Ao_B]\,{}^AR_B \;\vdots\; {}^AR_B \right] \begin{bmatrix} {}^B\omega \\ {}^Bv_{o_B} \end{bmatrix}$

②

$$\left.\begin{matrix} ① \\ ② \end{matrix}\right\} \Rightarrow {}^A\mathcal{V} = \begin{bmatrix} {}^A\omega \\ {}^Av_{o_A} \end{bmatrix} = \underbrace{\begin{bmatrix} \overset{3\times 3}{{}^AR_B} & 0 \\ [{}^Ao_B]\,{}^AR_B & {}^AR_B \end{bmatrix}}_{6\times 6} \underbrace{\begin{bmatrix} {}^B\omega \\ {}^Bv_{o_B} \end{bmatrix}}_{6\times 1} \begin{matrix} 3\times 1 \\ 3\times 1 \end{matrix}$$

$\triangleq {}^AX_B \Longleftrightarrow$ change of cordinate matrix for twist

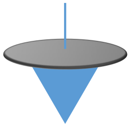
depends: on conf of B relative to A ${}^AT_B = ({}^AR_B, {}^Ao_B)$

- If configuration $\{B\}$ in $\{A\}$ is $T = (R, p)$, then

$$\boxed{{}^AX_B = [\mathrm{Ad}_T] \triangleq \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}}$$

Adjoint operator.

# Example I Revisited



# Outline

- Rigid Body Configuration
- Rigid Body Velocity (Twist)
- Geometric Aspect of Twist: Screw Motion

direction

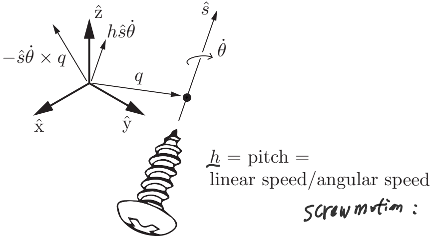
- Recall: linear velocity $v \in \mathbb{R}^3$ , $v = \hat{v} \cdot \|v\| \leftarrow$ scalar
- angular velocity: $\omega \in \mathbb{R}^3$. $\omega = \hat{\omega}\dot{\theta} \leftrightarrow$ scalar speed.

direction

- rigid body velocity: $\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} \overset{?}{=}$ Direction · speed

# Screw Motion: Definition

- Rotating about an axis while also translating along the axis



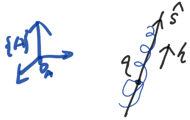

screw motion : $(q, \hat{s}, h, \dot{\theta})$

- Represented by screw axis $\{q, \hat{s}, h\}$ and rotation speed $\dot{\theta}$
  - $\hat{s}$: unit vector in the direction of the rotation axis
  - $q$: any point on the rotation axis
  - $h$: **screw pitch** which defines the ratio of the linear velocity along the screw axis to the angular velocity about the screw axis

- Theorem (Chasles): Every rigid body motion can be realized by a screw motion.

# From Screw Motion to Twist : screw motion is a special rigid body motion

- Consider a rigid body under a screw motion with screw axis $\{\hat{s}, h, q\}$ and (rotation) speed $\dot{\theta}$
- Fix a reference frame {A} with origin $o_A$.
- Find the twist ${}^A\mathcal{V} = ({}^A\omega, {}^A v_{o_A})$




$${}^A\omega = {}^A\hat{s}\cdot\dot{\theta}$$

$${}^A v_{o_A} = {}^A\hat{s}\, h\dot{\theta} + {}^A\omega \times (-{}^A q)$$

$$= {}^A\hat{s}(h\dot{\theta}) - {}^A\hat{s}\dot{\theta} \times {}^A q$$

$$= ({}^A\hat{s}\, h - {}^A\hat{s} \times {}^A q)\cdot\dot{\theta}$$

{A}

"Coordinate" free" given arbitrary body-fixed point p. using q as ref pt

$$v_p = v_q + \omega \times (\vec{qp})$$

$$v_{o_A} = v_q + \omega \times (\vec{q o_A})$$

$$= \hat{s}\dot{\theta} h + \omega \times (\vec{q o_A})$$

- **Result**: given screw axis $\{\hat{s}, h, q\}$ with rotation speed $\dot{\theta}$, the corresponding twist $\mathcal{V} = (\omega, v)$ is given by

$$\omega = \hat{s}\dot{\theta} \qquad v = -\hat{s}\dot{\theta} \times q + h\hat{s}\dot{\theta}$$

  - The result holds as long as all the vectors and the twist are represented in the same reference frame

screw axis

$${}^A\mathcal{V} = \text{ScrewToTwist}(\hat{s}, h, q, \dot{\theta}) = \text{ScrewToTwist}(\hat{s}, h, q, \dot{\theta}=1)\cdot\dot{\theta}$$

# From Twist to Screw Motion

- The converse is true as well: given any twist $\mathcal{V} = (\omega, v)$ we can always find the corresponding screw motion $\{q, \hat{s}, h\}$ and $\theta$
  - If $\omega = 0$, then it is a pure translation ($h = \infty$)

$$\hat{s} = \frac{v}{\|v\|}, \quad \dot{\theta} = \|v\|, h = \infty, q \text{ can be arbitrary}$$

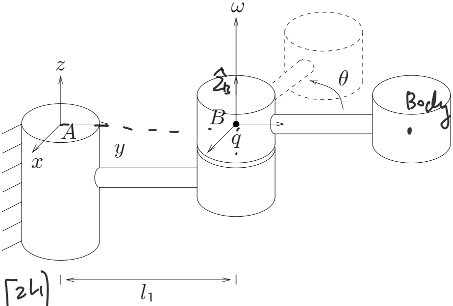
  - If $\omega \neq 0$:

$$\hat{s} = \frac{\omega}{\|\omega\|}, \quad \dot{\theta} = \|\omega\|, \quad q = \frac{\omega \times v}{\|\omega\|^2}, \quad h = \frac{\omega^T v}{\|\omega\|}$$

you can plug into the equation to verify the result.

# Examples: Screw Axis and Twist

- What is the twist that corresponds to rotating about $\hat{z}_B$ with $\dot{\theta} = 2$?




. choose $\{A\}$ frame.

$${}^A\hat{S} = {}^A\hat{z}_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad {}^Aq = \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix}, \quad h = 0, \quad \dot{\theta} = 2$$

$${}^A w = {}^A\hat{S}\,\dot{\theta} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$${}^A v_{0_A} = {}^A\hat{S}\, h\, \dot{\theta} - {}^A w \times {}^A q = 0 - \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \times \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2l_1 \\ 0 \\ 0 \end{bmatrix}$$

- What is the screw axis for twist $\mathcal{V} = \underbrace{(0, 2, 2, }_{w}4, 0, 0)$? $= \sqrt{8} \cdot \left( \frac{1}{\sqrt{8}} (\quad) \right)$

$$\hat{S} = \frac{\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}}{\sqrt{8}}, \quad \dot{\theta} = \sqrt{8}, \quad q = \frac{w \times v}{\|w\|^2}, \quad h =$$

# Screw Representation of a Twist

- Recall: an angular velocity vector $\omega$ can be viewed as $\hat{\omega}\dot{\theta}$, where $\hat{\omega}$ is the unit rotation axis and $\dot{\theta}$ is the rate of rotation about that axis

$\omega = \hat{\omega}\dot{\theta}$

- Similarly, a twist (spatial velocity) $\mathcal{V}$ can be interpreted in terms of a **screw axis** $\hat{\mathcal{S}}$ and a velocity $\dot{\theta}$ about the screw axis

$\mathcal{V} = \hat{S} \cdot \dot{\theta}$    $\hat{S} = \text{ScrewToTwist}(\hat{s}, q, h, \dot{\theta}=1)$

- Consider a rigid body motion along a screw axis $\hat{\mathcal{S}} = \{\hat{s}, h, q\}$ with speed $\dot{\theta}$. With slight abuse of notation, we will often write its twist as

$$\mathcal{V} = \hat{\mathcal{S}}\dot{\theta}$$

$\hat{S} \leftrightarrow (\hat{s}, h, q), \dot{\theta} = 1$

  - In this notation, we think of $\hat{\mathcal{S}}$ as the twist associated with a unit speed motion along the screw axis $\{\hat{s}, h, q\}$

# More Discussions