MEE5114 Advanced Control for Robotics

Lecture 6: Velocity Kinematics: Geometric and Analytic Jacobian of Open Chain

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Outline

- Background
- Geometric Jacobian Derivations
- Analytic Jacobian

Velocity Kinematics

- **Velocity Kinematics**: How does the velocity of $\{b\}$ relate to the joint velocities $\dot{\theta}_1, \ldots, \dot{\theta}_n$
- This depends on how to represent $\{b\}$'s velocity
  - Twist representation $\rightarrow$ **Geometric Jacobian**
  - Local coordinate of SE(3) $\rightarrow$ **Analytic Jacobian**

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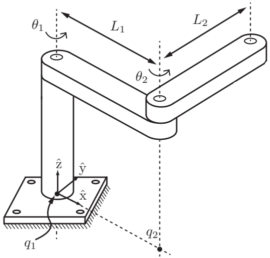
Simple Illustration Example: Geometric Jacobian (1/2)



Simple Illustration Example: Geometric Jacobian (2/2)

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Geometric Jacobian: General Case (1/3)

- Let $\mathcal{V} = (\omega, v)$ be the end-effector twist (coordinate-free notation), we aim to find $J(\theta)$ such that

$$\mathcal{V} = J(\theta)\dot{\theta} = J_1(\theta)\dot{\theta}_1 + \cdots + J_n(\theta)\dot{\theta}_n$$

- The ith column $J_i(\theta)$ is the end-effector *velocity* when the robot is rotating about $\mathcal{S}_i$ at unit speed $\dot{\theta}_i = 1$ while all other joints do not move (i.e. $\dot{\theta}_j = 0$ for $j \neq i$).

- Therefore, in **coordinate free** notation, J_i is just the screw axis of joint i:

$$J_i(\theta) = \mathcal{S}_i(\theta)$$

Geometric Jacobian: General Case (2/3)

- The actual coordinate of $\mathcal{S}_i$ depends on θ as well as the reference frame.

- The simplest way to write Jacobian is to use local coordinate:

$$^{i}J_i = {}^{i}S_i, \quad i = 1, \ldots, n$$

- In fixed frame $\{0\}$, we have

$$^{0}J_i(\theta) = {}^{0}X_i(\theta)\, {}^{i}S_i, \quad i = 1, \ldots, n \tag{1}$$

- Recall: ${}^{0}X_i$ is the change of coordinate matrix for spatial velocities.

- Assume $\theta = (\theta_1, \ldots, \theta_n)$, then

$$^{0}T_i(\theta) = e^{[{}^{0}\bar{\mathcal{S}}_1]\theta_1} \cdots e^{[{}^{0}\bar{\mathcal{S}}_i]\theta_i} M \quad \Rightarrow \quad {}^{0}X_i(\theta) = \left[\mathrm{Ad}_{{}^{0}T_i(\theta)}\right] \tag{2}$$

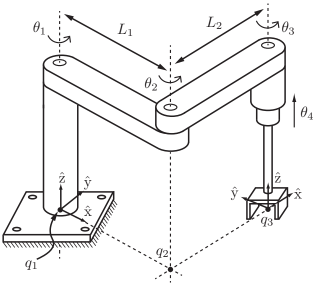
Geometric Jacobian: General Case (3/3)

- The Jacobian formula (1) with (2) is conceptually simple, but can be cumbersome for calculation. We now derive a recursive Jacobian formula

- Note: ${}^0J_i(\theta) = {}^0S_i(\theta)$
  - For $i = 1$, ${}^0S_1(\theta) = {}^0S_1(0) = {}^0\bar{\mathcal{S}}_1$ (independent of θ)
  - For $i = 2$, ${}^0S_2(\theta) = {}^0S_1(\theta_1) = \left[\mathrm{Ad}_{\hat{T}(\theta_1)}\right] {}^0\bar{\mathcal{S}}_2$, where $\hat{T}(\theta_1) \triangleq e^{[{}^0\bar{\mathcal{S}}_1]\theta_1}$
  - For general i, we have

$$
\begin{aligned}
&{}^0J_i(\theta) = {}^0S_i(\theta) = \left[\mathrm{Ad}_{\hat{T}(\theta_1,\ldots,\theta_{i-1})}\right] {}^0\bar{\mathcal{S}}_i \\
\text{where} \quad &\hat{T}(\theta_1,\ldots,\theta_{i-1}) \triangleq e^{[{}^0\bar{\mathcal{S}}_1]\theta_1} \cdots e^{[{}^0\bar{\mathcal{S}}_{i-1}]\theta_{i-1}}
\end{aligned}
\tag{3}
$$

Geometric Jacobian Example



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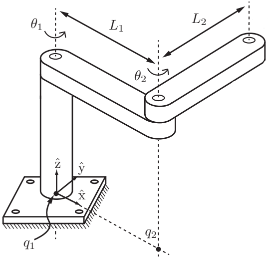
Analytic Jacobian

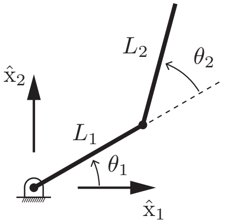
- Let $x \in \mathbb{R}^p$ be the task space variable of interest with desired reference x_d
  - E.g.: x can be Cartesian + Euler angle of end-effector frame
  - $p < 6$ is allowed, which means a partial parameterization of SE(3), e.g. we only care about the position or the orientation of the end-effector frame
- Analytic Jacobian: $\dot{x} = J_a(\theta)\dot{\theta}$
- Recall Geometric Jacobian: $\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} = J(\theta)\dot{\theta}$
- They are related by:

$$J_a(\theta) = E(x)J(\theta) = E(\theta)J(\theta)$$

  - $E(x)$ can be easily found with given parameterization x

Simple Illustration Example: Analytic Jacobian (1/3)





$$\begin{cases} {}^0p_{b,x} = L_1 \cos(\theta_1) + L_2 \cos(\theta_1+\theta_2) \\ {}^0p_{b,y} = L_1 \sin(\theta_1) + L_2 \sin(\theta_1+\theta_2) \\ {}^0p_{b,z} = L_0 \end{cases}$$

Simple Illustration Example: Analytic Jacobian (2/3)

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Simple Illustration Example: Analytic Jacobian (3/3)

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More Discussions

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