SDM5008 Advanced Control for Robotics

Lecture 2: Operator View of Rigid-Body Transformation

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Outline

- Matrix Exponential
- Rotation Operation via Differential Equation
- Rigid-Body Operation via Differential Equation
- Rigid-Body Operation of Screw Axis

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How to Solve Linear Differential Equations?

- Consider a scalar linear system: $z(t) \in \mathbb{R}$ and $a \in \mathbb{R}$ is a constant

$$\dot{z}(t) = az(t), \quad \text{with initial condition } z(0) = z_0 \tag{1}$$

- The above ODE has a unique solution:

- What about general linear systems? $\dot{x} = Ax + Bu$

What is the "Euler's Number" e?

- What is the number "e"?

Complex Exponential

- For real variable $x \in \mathbb{R}$, Taylor series expansion for e^x around $x = 0$:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

- This can be extended to complex variables:

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

  This power series is well defined for all $z \in \mathbb{C}$

- In particular, we have $e^{j\theta} = 1 + j\theta - \frac{\theta^2}{2} - j\frac{\theta^3}{3!} + \cdots$

- Comparing with Taylor expansions for $\cos(\theta)$ and $\sin(\theta)$ leads to the Euler's Formula

Matrix Exponential Definition

- Similar to the real and complex cases, we can define the so-called *matrix exponential*

$$e^A \triangleq \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

- This power series is well defined for any finite square matrix $A \in \mathbb{R}^{n \times n}$.

Some Important Properties of Matrix Exponential

- $Ae^A = e^A A$

- $e^A e^B = e^{A+B}$ if $AB = BA$

- If $A = PDP^{-1}$, then $e^A = Pe^D P^{-1}$

- For every $t, \tau \in \mathbb{R}$, $e^{At} e^{A\tau} = e^{A(t+\tau)}$

- $\left(e^A\right)^{-1} = e^{-A}$

Autonomous Linear Systems

$$\dot{x}(t) = Ax(t), \quad \text{with initial condition } x(0) = x_0 \tag{2}$$

- $x(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ is constant matrix, $x_0 \in \mathbb{R}^n$ is given.

- With the definition of matrix exponential, we can show that the solution to (2) is given by

$$x(t) = e^{At}x_0$$

Computation of Matrix Exponential

- Directly from definition

- For diagonalizable matrix:

- Using Padé approximation

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Skew Symmetric Matrices

- Recall that cross product is a special linear transformation.
- For any $\omega \in \mathbb{R}^n$, there is a matrix $[\omega] \in \mathbb{R}^{n \times n}$ such that $\omega \times p = [\omega]p$

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \leftrightarrow [\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

- Note that $[\omega] = -[\omega]^T \leftarrow$ skew symmetric
- $[\omega]$ is called a skew-symmetric matrix representation of the vector ω
- The set of skew-symmetric matrices in: $so(n) \triangleq \{S \in \mathbb{R}^{n \times n} : S^T = -S\}$
- We are interested in case $n = 2, 3$

Rotation Operation via Differential Equation

- Consider a point initially located at p_0 at time $t = 0$
- Rotate the point with unit angular velocity $\hat{\omega}$. Assuming the rotation axis passing through the origin, the motion is described by

$$\dot{p}(t) = \hat{\omega} \times p(t) = [\hat{\omega}]p(t), \text{ with } p(0) = p_0 \tag{3}$$

- This is a linear ODE with solution: $p(t) = e^{[\hat{\omega}]t} p_0$
- After $t = \theta$, the point has been rotated by θ degree. Note $p(\theta) = e^{[\hat{\omega}]\theta} p_0$
- $\mathrm{Rot}(\hat{\omega}, \theta) \triangleq e^{[\hat{\omega}]\theta}$ can be viewed as a rotation operator that rotates a point about $\hat{\omega}$ through θ degree

Rotation Matrix as a Rotation Operator (1/3)

- Every rotation matrix R can be written as $R = \mathrm{Rot}(\hat{\omega}, \theta) \triangleq e^{[\hat{\omega}]\theta}$, i.e., it represents a rotation operation about $\hat{\omega}$ by θ.

- We have seen how to use R to represent frame orientation and change of coordinate between different frames. They are quite different from the operator interpretation of R.

- To apply the rotation operation, all the vectors/matrices have to be expressed in the **same reference frame** (this is clear from Eq (3))

Rotation Matrix as a Rotation Operator (2/3)

- For example, assume $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \mathrm{Rot}(\hat{\mathrm{x}}; \pi/2)$
- Consider a relation $q = Rp$:
  - **Change reference frame interpretation** :
  - **Rotation operator interpretation**:

Rotation Matrix as a Rotation Operator (3/3)

- Consider the frame operation:
  - Change of reference frame: $R_B = RR_A$
  - Rotating a frame: $R'_A = RR_A$

Rotation Matrix Properties

- $R^T R = I$
- $R_1 R_1 \in SO(3)$, if $R_1, R_2 \in SO(3)$
- $\|Rp - Rq\| = \|p - q\|$
- $R(v \times w) = (Rv) \times (Rw)$
- $R[w]R^T = [Rw]$

Rotation Operator in Different Frames (1/2)

- Consider two frames {A} and {B}, the actual numerical values of the operator $\mathrm{Rot}(\hat{\omega}, \theta)$ depend on both the reference frame to represent $\hat{\omega}$ and the reference frame to represent the operator itself.

- Consider a rotation axis $\hat{\omega}$ (coordinate free vector), with {A}-frame coordinate ${}^A\hat{\omega}$ and {B}-frame coordinate ${}^B\hat{\omega}$. We know

$$^A\hat{\omega} = {}^AR_B\, {}^B\hat{\omega}$$

- Let ${}^B\mathrm{Rot}({}^B\hat{\omega}, \theta)$ and ${}^A\mathrm{Rot}({}^A\hat{\omega}, \theta)$ be the two rotation matrices, representing the same rotation operation $\mathrm{Rot}(\hat{\omega}, \theta)$ in frames {A} and {B}.

Rotation Operator in Different Frames (2/2)

- We have the relation:

$$ {}^{A}\mathrm{Rot}({}^{A}\hat{\omega}, \theta) = {}^{A}R_{B}\, {}^{B}\mathrm{Rot}({}^{B}\hat{\omega}, \theta)\, {}^{B}R_{A} $$

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Rigid-Body Operation via Differential Equation (1/3)

- Recall: Every $R \in SO(3)$ can be viewed as the state transition matrix associated with the rotation ODE(3). It maps the initial position to the current position (after the rotation motion)

  - $p(\theta) = \mathrm{Rot}(\hat{\omega}, \theta)p_0$ viewed as a solution to $\dot{p}(t) = [\hat{\omega}]p(t)$ with $p(0) = p_0$ at $t = \theta$.

  - The above relation requires that the rotation axis passes through the origin.

- We can obtain similar ODE characterization for $T \in SE(3)$, which will lead to exponential coordinate of $SE(3)$

Rigid-Body Operation via Differential Equation (2/3)

- Recall: Theorem (Chasles): Every rigid body motion can be realized by a screw motion

- Consider a point p undergoes a screw motion with screw axis $\mathcal{S}$ and unit speed ($\dot{\theta} = 1$). Let the corresponding twist be $\mathcal{V} = \mathcal{S} = (\omega, v)$. The motion can be described by the following ODE.

$$\dot{p}(t) = \omega \times p(t) + v \quad \Rightarrow \quad \begin{bmatrix} \dot{p}(t) \\ 0 \end{bmatrix} = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ 1 \end{bmatrix} \qquad (4)$$

- Solution to (4) in homogeneous coordinate is:

$$\begin{bmatrix} p(t) \\ 1 \end{bmatrix} = \exp\left(\begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} t \right) \begin{bmatrix} p(0) \\ 1 \end{bmatrix}$$

Rigid-Body Operation via Differential Equation (3/3)

- For any twist $\mathcal{V} = (\omega, v)$, let $[\mathcal{V}]$ be its matrix representation

$$[\mathcal{V}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}$$

- The above definition also applies to a screw axis $\mathcal{S} = (\omega, v)$

- With this notation, the solution to (4) is $\tilde{p}(t) = e^{[\mathcal{S}]t}\tilde{p}(0)$

- Fact: $e^{[\mathcal{S}]t} \in SE(3)$ is always a valid homogeneous transformation matrix.

- Fact: Any $T \in SE(3)$ can be written as $T = e^{[\mathcal{S}]t}$, i.e., it can be viewed as an operator that moves a point/frame along the screw axis at unit speed for time t

$se(3)$

- Similar to $so(3)$, we can define $se(3)$:

$$se(3) = \{([\omega], v) : [\omega] \in so(3), v \in \mathbb{R}^3\}$$

- $se(3)$ contains all matrix representation of twists or equivalently all twists.

- In some references, $[\mathcal{V}]$ is called a twist.

- Sometimes, we may abuse notation by writing $\mathcal{V} \in se(3)$.

Homogeneous Transformation as Rigid-Body Operator

- ODE for rigid motion under $\mathcal{V} = (\omega, v)$

$$\dot{p} = v + \omega \times p \quad \Rightarrow \dot{\tilde{p}}(t) = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \tilde{p}(t) \Rightarrow \tilde{p}(t) = e^{[\mathcal{V}]t} \tilde{p}(0)$$

- Consider "unit velocity" $\mathcal{V} = \mathcal{S}$, then time t means degree

- $\tilde{p}' = T\tilde{p}$: "rotate" p about screw axis $\mathcal{S}$ by θ degree

- TT_A: "rotate" {A}-frame about $\mathcal{S}$ by θ degree

Rigid-Body Operator in Different Frames

- Expression of T in another frame (other than {O}):

$$\underset{\text{operation in } \{O\}}{T} \quad \leftrightarrow \quad \underset{\text{operation in } \{B\}}{T_B^{-1} T T_B}$$

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Rigid Operation on Screw Axis

- Consider an arbitrary screw axis $\mathcal{S}$, suppose the axis has gone through a rigid transformation $T = (R, p)$ and the resulting new screw axis is $\mathcal{S}'$, then

$$\mathcal{S}' = [\mathrm{Ad}_T]\,\mathcal{S}$$

proof:

More Space