SDM5008 Advanced Control for Robotics

Lecture 1: Rigid Body Configuration and Velocity

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Outline

- Rigid Body Configuration
- Rigid Body Velocity (Twist)
- Geometric Aspect of Twist: Screw Motion

Free Vector

- **Free Vector**: geometric quantity with length and direction

- Given a reference frame, v can be moved to a position such that the base of the arrow is at the origin without changing the orientation. Then the vector v can be represented by its coordinates v in the reference frame.

- v denotes the physical quantity while ${}^{A}v$ denote its coordinate wrt frame $\{$A$\}$.

Point

- **Point**: p denotes a point in the physical space

- A point p can be represented by a vector from frame origin to p

- ${}^{A}p$ denotes the coordinate of a point p wrt frame $\{\mathsf{A}\}$

- When left-superscript is not present, it means the physical vector itself or the coordinate of the vector for which the reference frame is clear from the context.

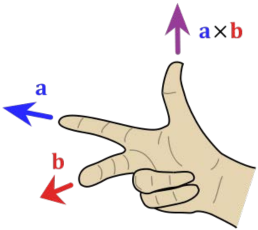
Cross Product

- **Cross product** or **vector product** of $a \in \mathbb{R}^3, b \in \mathbb{R}^3$ is defined as

$$a \times b = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} \tag{1}$$

Properties:

- $\|a \times b\| = \|a\|\|b\| \sin(\theta)$
- $a \times b = -b \times a$
- $a \times a = 0$



Skew symmetric representation

- It can be directly verified from definition that $a \times b = [a]b$, where

$$[a] \triangleq \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{2}$$

- $a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \leftrightarrow [a]$

- $[a] = -[a]^T$ (called skew symmetric)

- $[a][b] - [b][a] = [a \times b]$ (Jacobi's identity)

Rotation Matrix

- **Frame**: 3 coordinate vectors (unit length) $\hat{x}, \hat{y}, \hat{z}$, and an origin
  - $\hat{\mathrm{x}}, \hat{\mathrm{y}}, \hat{\mathrm{z}}$ mutually orthogonal
  - $\hat{\mathrm{x}} \times \hat{\mathrm{y}} = \hat{\mathrm{z}}$
- **Rotation Matrix**: specifies orientation of one frame relative to another

$$^{A}R_{B} = \begin{bmatrix} ^{A}\hat{x}_{B} & ^{A}\hat{y}_{B} & ^{A}\hat{z}_{B} \end{bmatrix}$$

- A valid rotation matrix R satisfies: (i) $R^T R = I$; (ii) $\det(R) = 1$

Special Orthogonal Group

- **Special Orthogonal Group**: Space of Rotation Matrices in $\mathbb{R}^n$ is defined as

$$SO(n) = \{R \in \mathbb{R}^{n \times n} : R^T R = I, \det(R) = 1\}$$

- $SO(n)$ is a *group*. We are primarily interested in $SO(3)$ and $SO(2)$, rotation groups of $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively.

- **Group** is a set G, together with an operation $\bullet$, satisfying the following group axioms:
  - **Closure:** $a \in G, b \in G \Rightarrow a \bullet b \in G$
  - **Associativity:** $(a \bullet b) \bullet c = a \bullet (b \bullet c)$, $\forall a, b, c \in G$
  - **Identity element:** $\exists e \in G$ such that $e \bullet a = a$, for all $a \in G$.
  - **Inverse element:** For each $a \in G$, there is a $b \in G$ such that $a \bullet b = b \bullet a = e$, where e is the identity element.

Use of Rotation Matrix (1/2)

- Representing an orientation ${}^{A}R_{B}$

- Changing the reference frame ${}^{A}R_{B}$:

Use of Rotation Matrix (2/2)

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- Rotating a vector or a frame $\mathrm{Rot}(\hat{\omega}, \theta)$: will be discussed in next lecture.

Rigid Body Configuration

- Given two coordinate frames {A} and {B}, the configuration of B relative to A is determined by
  - ${}^{A}R_{B}$ and ${}^{A}o_{B}$

- For a (free) vector r, its coordinates ${}^{A}r$ and ${}^{B}r$ are related by:

- For a point p, its coordinates ${}^{A}p$ and ${}^{B}p$ are related by:

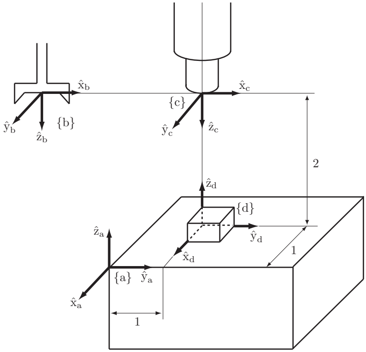
Homogeneous Transformation Matrix

- **Homogeneous Transformation Matrix**: ${}^{A}T_{B}$

- Homogeneous coordinates:

Example of Homogeneous Transformation Matrix

Fixed frame $\{a\}$; end effector frame $\{b\}$, the camera frame $\{c\}$, and the workpiece frame $\{d\}$. Suppose $\|p_c - p_b\| = 4$



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Rigid Body Velocity (1/3)

- Consider a rigid body in motion. The body has infinitely many points $\{p_i\}$ with different velocities $\{v_{p_i}\}$

  - All these velocities v_{p_i}'s are not independent
  - They can be expressed by the same set of parameter
  - Rigid body velocity (i.e. spatial velocity, twist) is one such parameterization

Rigid Body Velocity (2/3)

- Pure rotation case

- General motion

Rigid Body Velocity (3/3)

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Rigid Body Velocity: Spatial Velocity (Twist)

- How to represent a rigid body velocity?
  - Pick an arbitrary point r (reference point), which may or may not be body-fixed
  - Define v_r as the velocity of **the body-fixed point currently coincides with** r
  - For any body-fixed point p on the body: $v_p = v_r + \omega \times (\overrightarrow{rp})$
- **Spatial Velocity (Twist)**: $\mathcal{V}_r = (\omega, v_r)$
- Twist is a "physical" quantity (just like linear or angular velocity)
  - It can be represented in any frame for any chosen reference point r
- A rigid body with $\mathcal{V}_r = (\omega, v_r)$ can be "thought of" as translating at v_r while rotating with angular velocity ω about an axis passing through r
  - This is just one way to interpret the motion.

Spatial Velocity Representation in a Reference Frame

- Given frame {A} and a spatial velocity $\mathcal{V}$

- Choose o_A (the origin of {A}) as the reference point to represent the rigid body velocity

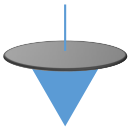
- Coordinates of $\mathcal{V}$ in {A}:

$$^A\mathcal{V}_{o_A} = (^A\omega, {}^A v_{o_A})$$

- By default, we assume the origin of the frame is used as the reference point: $^A\mathcal{V} = {}^A\mathcal{V}_{o_A}$

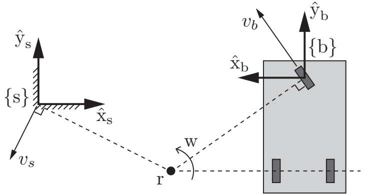
Example of Twist I

- Example I:



Example of Twist II

- Example II:



$r_s = (2, -1, 0)$, $r_b = (2, -1.4, 0)$, w=2 rad/s

Change Reference Frame for Twist (1/2)

- Given a twist $\mathcal{V}$, let ${}^A\mathcal{V}$ and ${}^B\mathcal{V}$ be their coordinates in frames $\{A\}$ and $\{B\}$

$$ {}^A\mathcal{V} = \begin{bmatrix} {}^A\omega \\ {}^A v_A \end{bmatrix}, \qquad {}^B\mathcal{V} = \begin{bmatrix} {}^B\omega \\ {}^B v_B \end{bmatrix} $$

- They are related by ${}^A\mathcal{V} = {}^A X_B \, {}^B\mathcal{V}$

Change Reference Frame for Twist (2/2)

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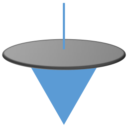
- If configuration {B} in {A} is $T = (R, p)$, then

$$^{A}X_B = [\mathrm{Ad}_T] \triangleq \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}$$

Example I Revisited



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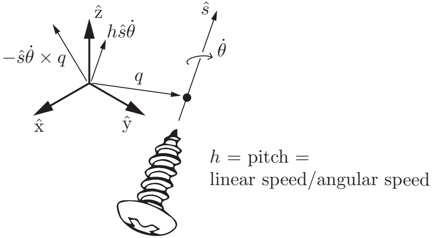
Screw Motion: Definition

- Rotating about an axis while also translating along the axis



- Represented by screw axis $\{q, \hat{s}, h\}$ and rotation speed $\dot{\theta}$
  - $\hat{s}$: unit vector in the direction of the rotation axis
  - q: any point on the rotation axis
  - h: **screw pitch** which defines the ratio of the linear velocity along the screw axis to the angular velocity about the screw axis

- Theorem (Chasles): Every rigid body motion can be realized by a screw motion.

From Screw Motion to Twist

- Consider a rigid body under a screw motion with screw axis $\{\hat{s}, h, q\}$ and (rotation) speed $\dot{\theta}$

- Fix a reference frame {A} with origin o_A.

- Find the twist ${}^A\mathcal{V} = ({}^A\omega, {}^A v_{o_A})$

- **Result**: given screw axis $\{\hat{s}, h, q\}$ with rotation speed $\dot{\theta}$, the corresponding twist $\mathcal{V} = (\omega, v)$ is given by

$$\omega = \hat{s}\dot{\theta} \qquad v = -\hat{s}\dot{\theta} \times q + h\hat{s}\dot{\theta}$$

  - The result holds as long as all the vectors and the twist are represented in the same reference frame

From Twist to Screw Motion

- The converse is true as well: given any twist $\mathcal{V} = (\omega, v)$ we can always find the corresponding screw motion $\{q, \hat{s}, h\}$ and $\dot{\theta}$
  - If $\omega = 0$, then it is a pure translation ($h = \infty$)

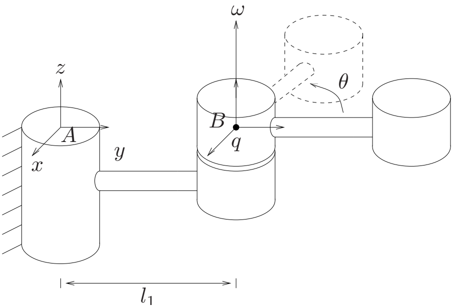
$$\hat{s} = \frac{v}{\|v\|}, \quad \dot{\theta} = \|v\|, h = \infty, q \text{ can be arbitrary}$$

  - If $\omega \neq 0$:

$$\hat{s} = \frac{\omega}{\|\omega\|}, \quad \dot{\theta} = \|\omega\|, \quad q = \frac{\omega \times v}{\|\omega\|^2}, \quad h = \frac{\omega^T v}{\|\omega\|}$$

Examples: Screw Axis and Twist

- What is the twist that corresponds to rotating about $\hat{z}_B$ with $\dot{\theta} = 2$?



- What is the screw axis for twist $\mathcal{V} = (0, 2, 2, 4, 0, 0)$?

Screw Representation of a Twist

- Recall: an angular velocity vector ω can be viewed as $\hat{\omega}\dot{\theta}$, where $\hat{\omega}$ is the unit rotation axis and $\dot{\theta}$ is the rate of rotation about that axis

- Similarly, a twist (spatial velocity) $\mathcal{V}$ can be interpreted in terms of a **screw axis** $\hat{\mathcal{S}}$ and a velocity $\dot{\theta}$ about the screw axis

- Consider a rigid body motion along a screw axis $\hat{\mathcal{S}} = \{\hat{s}, h, q\}$ with speed $\dot{\theta}$. With slight abuse of notation, we will often write its twist as

$$\mathcal{V} = \hat{\mathcal{S}}\dot{\theta}$$

  - In this notation, we think of $\hat{\mathcal{S}}$ as the twist associated with a unit speed motion along the screw axis $\{\hat{s}, h, q\}$

More Discussions